PHYS. FORMULAE – August 28, 2014

SI Base Units

Length L/metre: path length of light in vacuum during $dt = {}^1/_{299792458}s$. **Mass** M/kilogram: mass of an international prototype. **Time** T/second: dt of 9192631770 ^{133}Cs (hyperfine split, @$0\,K$) ground state oscillations. **Elec. Current** I/ampere: produces 2E−7 N/m between two $L=\infty$, $A=0$ wires $1\,m$ apart in vaccum. **Thermo. Temp.** Θ/kelvin: 1/273.16$^{\text{th}}$ the T of triple point water. **Amount of Substance** N/mole: as many entities (atoms, molecules, e^-, etc.) as there are atoms in $12\,g$ of ground state ^{12}C. **Luminous Intensity** J/candela: directional intensity of a monochromatic ($f = 5.40\text{E}12$ Hz) source with directional radiant intensity 1/683 watt per steradian.

Constants & Units

	Value(s)		Base
m_{e^-}	9.11E−31 kg	$(5.48\text{E}{-4})\cdot u$	M
m_{p^-}	1.67E−27 kg	$(1.01)\cdot u$	M
$\overline{m}_{air}$	$(29.0)\cdot u$	$\approx \frac{3}{4}\overset{m}{N_2} + \frac{1}{4}\overset{m}{O_2}$	M
$m_\otimes$	5.97E24 kg	$(3.60\text{E}51)\cdot u$	M
$m_\odot$	1.99E30 kg	$(3.33E5)\cdot m_\otimes$	M
k_B	1.38E−23 J/K	8.62E−5 eV/K	$ML^2/T^2\Theta$
R	8.31 J/mol K	$N_A k_B$	$ML^2/T^2\Theta N$
N_A	6.02E23 1/mol	$\approx 2^{79}$ 1/mol	1/N
h	6.23E−34 J·s	4.14E−15 eV·s	ML^2/T
e	1.60E−19 C	1 eV	IT
α	$\approx {}^1/137.036$	$e^2/4\pi\epsilon_0\hbar c$	–
μ_B	5.79E−5 eV/T	$e\hbar/2m_e$	IL^2
G	6.67E−11 Nm^2/kg^2	–	L^3/T^2M
σ_B	5.67E−8 J/sm^2K^4	$\pi^2k_B/60c^2\hbar^3$	
$1\,N$	$1\,kg\,m/s^2$	$1(Pa)m^2$	ML/T^2
$1\,J$	$1\,kg\,m^2/s^2$	$1N\,m, 1C\,V$	ML^2/T^2
$1\,W$	$1\,kg\,m^2/s^3$	$1J/s, 1V\,A, 1\Omega A^2$	ML^2/T^3
$1\,F$	$1\,s^4A^2/m^2kg$	$1\frac{J}{V^2}, 1\frac{C^2}{J}, 1\frac{s^2}{H}$	T^4I^2/L^2M
$1\,\Omega$	$1\,kg\,m^2/s^3A^2$	$1\frac{V}{A}, 1\frac{J\,s}{C^2}, 1\frac{s}{F}$	ML^2/T^3I^2
$1\,V$	$1\,kg\,m^2/s^3A$	$1\Omega A, 1J/C, 1W/A$	ML^2/T^3I
$1\,T$	$1\,kg/s^2A$	$1\frac{J}{m^2A}, 1\frac{kg}{C\,s}$	M/T^2I
$1\,H$	$1\,kg\,m^2/s^2A^2$	$1\frac{J}{A^2}, 1\Omega s, 1\frac{m^2T}{A}$	ML^2/T^2I^2

Math

Shape	Def. As	Circumference	Area
Circle	$x^2+y^2=r^2$	$2\pi r$	πr^2
Ellipse	$x^2/a^2+y^2/b^2=1$	$\approx\pi[3a+3b-\sqrt{(3a+b)(a+3b)}]$	πab

Shape	n-Area	n-Vol.
3-sphere	$4\pi r^2$	$4\pi r^3/3$
cone$_\perp$	$\pi r^2[1+\sqrt{1+(h/r)^2}]$	$\pi hr^2/3$
pyramid$_\perp$	$\frac{ab}{2}[2+\sqrt{1+(2h/a)^2}+\sqrt{1+(2h/b)^2}]$	$abh/3$
n-sphere	$2\pi^{n/2}r^{n-1}/\Gamma(n/2)$	$\pi^{n/2}r^n/\Gamma(1+n/2)$

Trigonometry Identities

$\sin(x)=(e^{ix}-e^{-ix})/2i$, $\cos(x)=(e^{ix}+e^{-ix})/2$ • $\sin(2x)=2\sin(x)\cos(x)$, $\cos(2x)=1-2\sin^2(x)$ • $\sinh(x)=(e^x-e^{-x})/2$, $\cosh(x)=(e^x+e^{-x})/2$

$\sin^2(x)=(1-\cos(2x))/2$, $\cos^2(x)=(1+\cos(2x))/2$ • $\sin(x\pm y)=\sin(x)\cos(y)\pm\cos(x)\sin(y)$, $\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$

$\cosh^2(x)-\sinh^2(x)=1$

Series and Sums

$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x$, $\lim_{x\to\infty}\left(1+\frac{x}{n}\right)^n=\left(\frac{x}{n}\right)^n$ • $\ln(1+x)=\begin{cases}\sum_{k=1}^\infty\frac{(-1)^{k+1}x^k}{k} & -1<x\le1\\ \ln(x)+\sum_{k=1}^\infty\frac{(-1)^{k+1}}{kx^k} & |x|>1\end{cases}$

$e^z=\sum_{n=0}^\infty\frac{z^n}{n!}$ • $\sum_{n=0}^N x^n=\frac{1-x^{N-1}}{1-x}\left(\xRightarrow[N\to\infty]{\text{if}}\frac{1}{1-x}\right)\Big|\,|x|<1$

Coordinates, Vector Calculus

$\int_{\vec a}^{\vec b}(\vec\nabla f)\cdot d\vec l=f(\vec b)-f(\vec a)$, $\vec\nabla(\hat{\mathscr{r}}/\mathscr{r}^2)=4\pi\delta^{(3)}(\vec{\mathscr{r}})$ • $\int\vec\nabla\cdot\vec A\,dV=\oint\vec A\cdot d\vec a$, $\nabla^2(1/\mathscr{r})=-4\pi\delta^{(3)}(\vec{\mathscr{r}})$ • $\int\vec\nabla\times\vec A\cdot d\vec a=\oint\vec A\cdot d\vec l$

$\vec A\cdot(\vec B\times\vec C)=\vec B\cdot(\vec C\times\vec A)=\vec C\cdot(\vec A\times\vec B)$, $\vec A\times(\vec B\times\vec C)=(\vec A\cdot\vec C)\vec B-(\vec A\cdot\vec B)\vec C$ • $\vec\nabla\cdot(\vec\nabla\times\vec A)=0=\vec\nabla\times(\vec\nabla f)$, $\vec\nabla\times(\vec\nabla\times\vec A)=\vec\nabla(\vec\nabla\cdot\vec A)-\nabla^2\vec A$

Spherical coords, $dV=r^2\sin\theta\,dr\,d\theta\,d\phi$ • $\vec\nabla t=\partial_r[t]\hat r+\frac{1}{r}\partial_\theta[t]\hat\theta+\frac{1}{r\sin\theta}\partial_\phi[t]\hat\phi$, $d\vec l=dr\hat r+rd\theta\hat\theta+r\sin\theta\,d\phi\hat\phi$

$\nabla^2t=\frac{1}{r^2}\partial_r\left[r^2\partial_r[t]\right]+\frac{1}{r^2\sin\theta}\partial_\theta\left[\sin\theta\partial_\theta[t]\right]+\frac{1}{r^2\sin^2\theta}\partial_\phi^2[t]$

$\vec\nabla\cdot\vec v=\frac{1}{r^2}\partial_r\left[r^2v_r\right]+\frac{1}{r\sin\theta}\partial_\theta\left[\sin\theta v_\theta\right]+\frac{1}{r\sin\theta}\partial_\phi[v_\phi]$

$x=rc\phi s\theta$, $y=rs\phi s\theta$, $z=rc\theta$, $\phi=\text{atan}(y/x)$ • $\hat x=s\theta c\phi\hat r+c\theta c\phi\hat\theta-s\phi\hat\phi$, $\hat y=s\theta s\phi\hat r+c\theta s\phi\hat\theta+c\phi\hat\phi$, $\hat z=c\theta\hat r-s\theta\hat\theta$, $\theta=\text{atan}(\sqrt{x\cdot x+y\cdot y}/z)$ • $\hat r=s\theta c\phi\hat x+s\theta s\phi\hat y+c\theta\hat z$, $\hat\theta=c\theta c\phi\hat x+c\theta s\phi\hat y-s\theta\hat z$, $\hat\phi=-s\phi\hat x+c\phi\hat y$, $r=\sqrt{x\cdot x+y\cdot y+z\cdot z}$

Cylindrical coords, $dV=r\,dr\,d\phi\,dz$ • $\vec\nabla t=\partial_r[t]\hat r+\frac{1}{r}\partial_\phi[t]\hat\phi+\partial_z[t]\hat z$, $d\vec l=dr\hat r+rd\phi\hat\phi+dz\hat z$

$\nabla^2t=\frac{1}{r}\partial_r[r\partial_r[t]]+\frac{1}{r^2}\partial_\phi^2[t]+\partial_z^2[t]$, $\vec\nabla\cdot\vec v=\frac{1}{r}\partial_r[rv_r]+\frac{1}{r}\partial_\phi[v_\phi]+\partial_z[v_z]$ • $\cancel{\partial_z^2}$ for polar/planar

$x=rc\phi$, $y=rs\phi$ • $z=z$, $\hat z=\hat z$ • $\hat x=c\phi\hat r-s\phi\hat\phi$, $\hat y=s\phi\hat r+c\phi\hat\phi$ • $\phi=\text{atan}(y/x)$, $r=\sqrt{x\cdot x+y\cdot y}$ • $\hat r=c\phi\hat x+s\phi\hat y$, $\hat\phi=-s\phi\hat x+c\phi\hat y$

Delta Function/Distrib.

$\delta^d(\vec r)=\frac{1}{(2\pi)^d}\int e^{i\vec k\cdot\vec r}d^dk$

Transforms

$f(\vec r)=\frac{1}{(2\pi)^{d/2}}\int_{-\infty}^\infty F(\vec k)e^{i\vec k\cdot\vec r}d^d\vec k$ • $F(\vec k)=\frac{1}{(2\pi)^{d/2}}\int_{-\infty}^\infty f(\vec r)e^{-i\vec k\cdot\vec r}d^d\vec r$

Combinatorics

$\binom{N}{n}=\frac{\overset{\text{(remaining possible)}}{N\cdot(N-1)\cdots(N-n+1)}}{\underset{\text{(order unimportant)}}{n!\ \hookleftarrow}}=\overset{\text{N choose n}}{\frac{N!}{n!\,(N-n)!}}$ • $\sum_{n=0}^N\binom{N}{n}=2^N$.

Stirling's Approx.

$n!\approx n^ne^{-n}\sqrt{2\pi n}\,(1+{}^1/12n)\quad\Longleftrightarrow\quad n\gg1$

$n!\approx n^ne^{-n}\sqrt{2\pi n}$ (sqrt sometimes needed)

$n!\sim n^ne^{-n}\quad\Longrightarrow\quad\ln n!\approx n\ln(n)-n$

Integrals – $(a,b>0)$

$\int_a^b\frac{1}{x}\sqrt{(x-a)(b-x)}\,dx=\frac{\pi}{2}(\sqrt b-\sqrt a)^2$ • $\int_0^\pi\sin^{2n}(x)\cos^{2m}(x)\,dx=\beta(n+\frac12,m+\frac12)$

$\int_0^\infty x^ne^{-bx}dx=n!/b^{n+1}\,\big|\,n\ge0$ • $\int_0^\infty x^ne^{-bx^2}dx=\frac12\frac{\Gamma((n+1)/2)}{b^{(n+1)/2}}\,\big|\,n\ge0$

$\int_0^\infty\frac{x^n}{e^x-1}dx=\Gamma(n+1)\zeta(n+1)\,\big|\,n\ge1$ • $\int\frac{1}{e^x+1}dx=-\ln(1+e^{-x})$

$\int_0^\infty\frac{x^n}{e^x+1}dx=(1-1/2^n)\Gamma(n+1)\zeta(n+1)\,\big|\,n\ge1$ • $\int_{-\infty}^\infty e^{-ax^2}dx=\frac12\sqrt{\frac{\pi}{a}}$

$\int_{-\infty}^\infty e^{-ax^2-2bx}dx=\frac12\sqrt{\frac{\pi}{a}}e^{b^2/a}$ • $\int_{-\infty}^\infty xe^{-a(x-b)^2}dx=b\sqrt{\frac{\pi}{a}}$

$\int_0^{2\pi}\int_0^\pi(\vec a\cdot\hat r)(\vec b\cdot\hat r)\sin\theta\,d\theta\,d\phi=4\pi(\vec a\cdot\vec b)/3$ • $\int_{-\infty}^\infty x^2e^{-ax^2}dx=\frac12\sqrt{\frac{\pi}{a^3}}$

$\int\frac{R\sin\theta\cos\theta\,d\theta}{(R^2+z^2-2zR\cos\theta)^{3/2}}=\frac{zR\cos\theta-R^2-z^2}{z^2\sqrt{R^2+z^2-2zR\cos\theta}}$ • $\int_a^b\sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}$

$\int_{-\infty}^\infty\frac{x^m\,dx}{(x^2+b^2)^n}=\pi\frac{\Gamma(n-q)\,|b|^{2(q-n)}}{\sin(\pi q)\Gamma(1-q)\Gamma(n)}\Big|\,q\equiv\frac{m+1}{2}$, $2n-m>1$, m even.

$\int\frac{dx}{(z^2+x^2)^{3/2}}=\frac{x}{z^2\sqrt{z^2+x^2}}$ • $\int\frac{dx}{\sqrt{z^2+x^2}}=\ln[x+\sqrt{z^2+x^2}]$

$\int_a^b\sqrt{\frac{b}{x}-1}\,dx=b\left(\frac{\pi}{2}-\arcsin\sqrt{\frac{a}{b}}\right)-\sqrt{a(b-a)}$ • $\int_0^\infty e^{-ar}\sin(br)\,dr=\frac{b}{b^2+a^2}$

$\int_0^\infty\frac{1}{r}\sin(\pm|b|r)\,dr=\pm\pi/2$

Misc

$\Gamma(1)=1$, $\Gamma(\frac12)=\sqrt\pi$ • $\Gamma(n)=(n-1)!$, $\zeta(2)=\pi^2/6$ • $\zeta(3)\approx1.202$, $\zeta(4)=\pi^4/90$ • $\beta(\frac12,\frac12)=\pi$, $\beta(\frac12,\frac32)=\frac{\pi}{2}$ • $\beta(\frac12,\frac52)=\frac{3\pi}{8}$, $\beta(\frac32,\frac32)=\frac{\pi}{8}$

$\Gamma(x)=(x-1)\Gamma(x-1)=\int_0^\infty t^{x-1}e^{-t}dt$ • $\beta(x,y)=\beta(y,x)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$

Classical Mech.

$\vec F=\frac{Gm_1m_2}{-\mathscr{r}^2}\hat{\mathscr{r}}$, $\vec L=\partial_t\vec\tau=\vec r\times\vec p$ • $\vec R=\frac{m_1\vec r_1+m_2\vec r_2}{m_1+m_2}$, $\vec\tau=\vec r\times\vec F$ • $\mu=\frac{m_1m_2}{m_1+m_2}$, $\vec r=\vec r_1-\vec r_2$ • $s^2=x^2+y^2$, $ds=\sqrt{1+(\partial_xy)^2}\,dx$

$\frac{\partial\mathcal L}{\partial q_i}=\frac{d}{dt}\frac{\partial\mathcal L}{\partial[\partial_tq_i]}$ Conserv. F's., • $\mathcal L=T-V=\frac{m}{2}(\partial_t\vec r)^2-V(\vec r)$ & Inertial Frm. • $T=\frac{m}{2}[(\partial_t\vec r)^2+r^2(\partial_t\phi)^2]$ Polar Kinetic.

Pendulum

Project $\vec F_g$ along $\hat T$ to cancel $(\hat T\perp\hat\theta)$. Now $F_{\text{net}}=-mg\sin\theta\approx-mg\theta$. So $ma=-mg\theta$. Since $s=L\theta$, solve $\ddot\theta=-\frac{g}{L}\theta\to\theta(t)=A\sin(\sqrt{g/L}\,t)+B\cos(\sqrt{g/L}\,t)$. Then, $f=\frac{\omega}{2\pi}=\frac{1}{2\omega}\sqrt{g/L}$ and $T=2\pi\sqrt{L/g}$.

Light Interference: $A\cos(kr+\omega t)$

Double slit: ∼ 2 point sources • $(I=2A)$max: $\sin\theta=n\lambda/d\Leftrightarrow n=0,\pm1,\pm2,..$; $(I=0)$min: $\sin\theta=n\lambda/2d\Leftrightarrow n=\pm1,\pm2,..$

Single slit: d = slit width • $(I=0)$min: $\sin\theta=n\lambda/d\Leftrightarrow n=\pm1,\pm2,..$; General: $I(\theta)=A(j_0(d\pi\sin(\theta)/\lambda))^2$

Special Relativity

$T=[\gamma(v)-1]mc^2$ • $\gamma(v)=1/\sqrt{1-v^2/c^2}$

$f_{obs}=f_{stat}/(\gamma(v)(1-\vec v\cdot\hat d/c))$ with $\hat v\cdot\hat d=\cos\theta$.

Scattering

Compton: (γ+○→γ′+♂) Inelastic, but ∼elastic.

1) E-cons.: $mc^2+\frac{hc}{\lambda}=\frac{hc}{\lambda'}+\sqrt{m^2c^4+p_{m'}^2c^2}$.

2) P-cons.: $\vec p_\gamma+\cancel{\vec p_m}=\vec p_{\gamma'}+\vec p_{m'}$. Isolate $\vec p_{m'}$, square.

Substitute for $\vec p_{m'}$: $\lambda'-\lambda=2\frac{h}{mc}\sin^2(\frac{\theta}{2})\,|\,\theta\leftarrow\vec p_\gamma\cdot\vec p_{\gamma'}$.

Thompson: Compton, $E_\gamma\ll mc^2$. (Elastic: ○→○).

Hard-sphere: $\frac{d\sigma}{d\Omega}=\frac{R^2}{4}\Rightarrow\sigma=\frac{R^2}{4}\int\sin\theta\,d\theta\,d\phi=\pi R^2$.

Rutherford: (♂+○→♂+○). Elastic, Heavy target, Coulomb. $\frac{d\sigma}{d\Omega}=\frac{(q_1q_2)^2}{(16\pi\epsilon_0T\sin^2\frac{\theta}{2})^2}\Big|\,T=mc^2(\gamma-1)$. (incoming kinetic)

Thermo./Stat. Mech.

1. $\Delta U=\overset{\text{heat}}{Q}+\overset{\text{work}}{W}$ • $W=\int_{V_i}^{V_f}\overset{\text{compression}}{P(V)dV}+\overset{\text{(other)}}{W_o}$

2. $S\equiv k_B\ln\Omega$ <u>grows</u>. Equilib. $\Rightarrow\frac{\partial S}{\partial\ldots}=0$ {N,U,V,...}.

3. $T\to0\Rightarrow C_V\to0$; $dS\rightsquigarrow0$ ($S\neq0$ b/c of $\Delta E=0$ config.s!)

$dU=TdS-PdV+\sum_i\mu_idN_i$. **Thermo. Identity.**

$\frac1T=\left(\frac{\partial S}{\partial U}\right)_{V,N_i}$ • $\frac{P}{T}=\left(\frac{\partial S}{\partial V}\right)_{U,N_i}$ • $\frac{\mu_i}{T}=-\left(\frac{\partial S}{\partial N_i}\right)_{U,V}$

$H=U+PV$. Enthalpy: E_{assemble} & put in enviro.

Reversible $dV\Rightarrow$ **quasistatic**. (Note: $\cancel{\Leftrightarrow}$!) • $Q\neq0\Rightarrow$ **irreversible**. But if $\Delta T\to0$ then $\Delta S\approx0$ so "infinitesimal" Q is ∼reversible. • **Quasistatic** $\Rightarrow dS=\frac{Q}{T}$; if $dV=dN=0$ then: $\Delta S=\int_{T_i}^{T_f}\frac{C_V}{T}dT$.

If $dN=0$ and $W=-PdV$ then: $\Delta S=\int_{T_i}^{T_f}\frac{C_P}{T}dT$.

Degrees of Freedom $\boxed{\text{f}}$

[Total at STP]=(trans, rot, spring{-vib, -pot}). Spring DOF are 'frozen out' in gases at STP. Monatomic gas [He, Ar] $\boxed{3}$=(3,0,{0,0}). Diatomic gas [O_2, N_2, CO] $\boxed{5}$=(3,2,{1,1}). Polyatomic (>2) gases: linear [CO_2] $\boxed{5}$=(3,2,{4,4}), nonlinear [NO_2, H_2O] $\boxed{6}$=(3,3,{3,3}). Elemental solids [Al, Pb] $\boxed{6}$=(0,0,{3,3}). Einstein solids $\boxed{2}$=(0,0,{1,1})/osc.. Adiabatic exponent: $\gamma\equiv1+2/f$.

Equipartition Theorem

At temp. T the avg. energy of *any* quadratic DOF is $\frac12k_BT$. For N entities with **only** f quadratic DOF each: $U_{\text{thermal}}=Nf\frac12k_BT$.

Ideal Gas

$PV=nRT=Nk_BT$ (Hard indist. spheres; low ρ_N; elastic coll's) • $U=Nf\frac12k_BT$ • $C_P=C_V+k_BN$.

$S(N,V,U)=Nk_B\left[\ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{\frac32}\right)+\frac52\right]\Big|$ monatomic.

$\mu=-k_BT\ln\left[\frac{V}{N}\left(\frac{4\pi mU}{3h^2N}\right)^{\frac32}\right]\cancel{+mgz}$ • $Z_{\text{tr}}(T,N=1)=V\left(\frac{2\pi m}{h^2\beta}\right)^{3/2}=\frac{V}{l_Q^3}=\frac{V}{v_Q}$

$Z(T,V,N)=\frac{1}{N!}[Z_1(T,V)]^N=\frac{1}{N!}[Z_{\text{int}}(T)Z_{\text{tr}}(T,N=1)]^N$

mixing gases $b\&c$

$\Delta S=-Nk_B\left(\frac{N_b}{N}\ln\left(\frac{N_b}{N}\right)+\frac{N_c}{N}\ln\left(\frac{N_c}{N}\right)\right)\Big|$ monatomic • $P_b=P_c$, $T_b=T_c$ • $N_b+N_c=N$

Free Expansion (Quasistatic+irreversible+vacuum) $\Delta U=\cancel{Q}+\cancel{W}=0$. But $\Delta S=Nk_B\ln\frac{V_f}{V_i}(\neq0,\neq\frac{Q}{T})$.

Isothermal (Quasistatic+slow+thermal equil.) $\Delta T=0\Rightarrow\cancel{\Delta U}=Q+W=0$ • $\partial(\frac{PV}{N})=0$ • $\Delta S=\frac{Q}{T}$

If $\Delta N=0$ then $W_{\text{comp.}}=Nk_BT\ln{}^{V_i}/_{V_f}$.

Adiabatic (Fast+~~quasistatic~~) $\Delta U=\cancel{Q}+W$ • $\Delta S\neq0$

If $\Delta N=0$ then $\partial(VT^{f/2})=0$, $\partial(V^\gamma P)=0$, and $W_{\text{comp.}}=\frac{f}{2}(P_fV_f-P_iV_i)$.

Isentropic (Adiabatic+Quasistatic) $\Delta S=\frac{\cancel{Q}}{T}=0$. Reversible.

$v_{\text{rms}}=\sqrt{\overline{v^2}}=\sqrt{3k_BT/\overline{m}}$ from toy piston/P_{wall}.

$c_{\text{sound}}=\sqrt{\frac{-V}{\rho_m}\frac{\partial P}{\partial V}}=\sqrt{\gamma k_BT/\overline{m}}$ [Adiabatic; expand $\partial(PV^\gamma)=0$, rearrange for bulk modulus.]

Heat (Energy) Capacity

$C\equiv\frac{Q}{\Delta T}=\frac{\Delta U-W}{\Delta T}$ • $c\equiv\frac{C}{m}$ • $C_V=\frac{\partial U}{\partial T}\Big|_{V,N,W}$

$C_P=\frac{\partial H}{\partial T}\Big|_P=\frac{\partial U}{\partial T}\Big|_P+P\frac{\partial V}{\partial T}\Big|_P-\frac{\partial W_o}{\partial T}\Big|_P$ • $L\equiv\frac{Q}{m}\Big|_{P,W}$

Virial Expansion / Van der Waals

$PV=nRT\,(1+{}^{B(T)}/_{(V/n)}+{}^{C(T)}/_{(V/n)^2}+\ldots)$

Qual. OK for gases, dense fluids. a : intermolecular attr., b : occupied V • $(P+an^2/V^2)(V-nb)=nRT$

2-State Paramagnet ∼ Spin 1/2

∼ Coins. • $\vec B\,||\,\hat z$ • $\Omega(N,N_\uparrow)\overset{\text{(symmetric!)}}{=}\binom{N}{N_\uparrow}$ • $\Omega_{\text{total}}=2^N$

$\Omega\approx\frac{(1+N_\downarrow/N_\uparrow)^{N_\uparrow}(1+N_\uparrow/N_\downarrow)^{N_\downarrow}}{\sqrt{2\pi N}}\Rightarrow\Omega\approx\left(\frac{eN}{N_\uparrow}\right)^{N_\uparrow}\Big|\,\overset{(N_\uparrow\leftrightarrows N_\downarrow\text{ ok})}{N_\downarrow\gg N_\uparrow\gg1}$

$S(U,B,N)=-Nk_B\left[\frac{1-z}{2}\ln\left(\frac{1-z}{2}\right)+\frac{1+z}{2}\ln\left(\frac{1+z}{2}\right)\right]$ • $\frac1T=\frac{k_B}{2\mu B}\ln\left(\frac{1-z}{1+z}\right)$

$U=\mu B(N_\downarrow-N_\uparrow)$, $M=\mu(N_\uparrow-N_\downarrow)$, $U(T,B)=-BM(T,B)$ • $z=U/(N\mu B)$ • $U(T,B)=-N\mu B\tanh(\mu B/k_BT)$, $C_B=Nk_B\left(\frac{\mu B}{k_BT}\right)^2/\cosh^2\left(\frac{\mu B}{k_BT}\right)$

Einstein Solid

N indep. oscil.'s, q quanta total: N bins & q stars.

Q■Q■Q■QQ → (1, 1, 1, 2)
Q■QQQ■Q■ → (1, 3, 1, 0)
. . . .
} ■ = bin dividers, $q+(N-1)$ symbols, choose q to be quanta.

$\Omega(N,q)=\binom{q+N-1}{q}$ • $E_{\text{\tiny WW}}=\hbar\omega(n+\frac12)$, $U=\hbar\omega(q+\cancel{\frac{N}{2}})$ • $V=\frac12k_sx^2$, $\omega=\sqrt{\frac{k_s}{m}}$

$\ln\Omega\approx\ln\left(1+\frac{N}{q}\right)^q+\ln\left(1+\frac{q}{N}\right)^N+\ldots$

$\Omega\approx\frac{(1+N/q)^q(1+q/N)^N}{\sqrt{2\pi q(q+N)/N}}\Big|\,q,N\gg1$ ⟨ $\Omega\approx\left(\frac{eq}{N}\right)^N\Big|\,\overset{\text{high-}T}{q\gg N\gg1}$; $\Omega\approx\left(\frac{eN}{q}\right)^q\Big|\,\underset{\text{low-}T}{N\gg q\gg1}$

$\mu\approx-k_BT\log(1+q/N)$.

High-T: $U\approx Nk_BT$ • $C_V\approx Nk_B$

Low-T: $U\approx\hbar\omega Ne^{-\hbar\omega/k_BT}$ • $C_V\approx Nk_B\left(\frac{\hbar\omega}{k_BT}\right)^2e^{-\hbar\omega/k_BT}$

Barometric Eqn.

Air slab: dz thick, $\overline{m}$ avg. molecule mass:

$\frac{dP}{dz}=\frac{N\overline{m}g}{-V}\overset{\text{ideal gas}}{\Longrightarrow}\frac{dP}{dz}=\frac{-\overline{m}g}{k_BT}P\Rightarrow N(z)=N_oe^{\frac{\overline{m}gz}{-k_BT}}$

Heat Eqn.

$\partial_tT=K\nabla^2T\,\big|\,K=\frac{mk_T}{\rho_mC}$ & $T(\vec r;0)=g(\vec r)$.

$T(\vec r;t)=\frac{1}{(4\pi Kt)^{n/2}}\int_{\mathbb R^n}e^{-\mathscr{r}^2/4Kt}g(\vec r')\,d\vec r'$

Partition Functions

Bltzmn/Can'cl $\mathcal P(s)=\frac1Z e^{-E(s)\beta}$ • Gibbs/Grnd can'cl $\mathcal P(r,N)=\frac{1}{\mathcal Z}e^{-(E(r,N)-\mu N)\beta}$ • $(r,N)\to$r$^{\text{th}}$ state of sys. of N particles.

$Z(T)=\sum_se^{-\beta E(s)}$ • $F(T)=-k_BT\ln(Z(T))$ • $\langle E\rangle=-\partial_\beta[\ln Z]$

$\langle Q\rangle=\sum_s\mathcal P_sQ(s)=\frac1Z\sum_sQ(s)e^{-\beta E(s)}$ • $\sigma_E^2=\partial_\beta^2[\ln Z]$

$\mathcal Z(T,\mu)=\sum_{(r,N)}e^{-(E(r,N)-\mu N)\beta}=\sum_N\left(e^{\beta\mu}\right)^NZ(T,V,N)$

Maxwell speed: $\mathcal D(v)dv=(4\pi v^2)\left(\frac{m\beta}{2\pi}\right)^{3/2}e^{-\beta mv^2/2}dv$

Most prob'bl: $v_p=\sqrt{\frac{2k_BT}{m}}$ • $\langle v\rangle=2v_p/\sqrt\pi$ • $v_{\text{rms}}=v_p\sqrt{3/2}$

Maxwell velocity: $\mathcal D^{(3)}(v)d^3\vec v=\left(\frac{m\beta}{2\pi}\right)^{3/2}e^{-\beta mv^2/2}d^3\vec v$

Ideal quantum gas $\mathcal Z(T,\mu)=\Pi_{i=1}^\infty\mathcal Z_i(T,V,\mu)$ • $\mathcal Z_i(T,\mu)=\mathcal Z(\epsilon_i,T,\mu)$, $\mathcal Z(\epsilon_i,T,\mu)=\sum_ne^{-\beta(\epsilon_i-\mu)n}$ • $\mathcal P(\epsilon_i,n)=\mathcal P_i(n)=\frac{e^{-\beta(\epsilon_i-\mu)n}}{\mathcal Z(\epsilon_i,T,\mu)}$

<u>Fermi-Dirac</u>: n=0,1 $\to\mathcal Z(\epsilon,T,\mu)=1+e^{-\beta(\epsilon-\mu)}\to\langle n\rangle(\epsilon,T,\mu)=\frac{1}{e^{+\beta(\epsilon-\mu)}+1}$

<u>Bose-Ein</u>: n=0,1,... $\to\mathcal Z(\epsilon,T,\mu)=\frac{1}{1-e^{-\beta(\epsilon-\mu)}}\to\langle n\rangle(\epsilon,T,\mu)=\frac{1}{e^{+\beta(\epsilon-\mu)}-1}$

$E(\vec k)=\frac{\hbar^2k^2}{2m}\Rightarrow k_x=\frac{2\pi n_x}{L}$ One part'cl in a box • $\sum_{n_x,n_y,n_z}\ldots=\sum_{\vec k}\ldots\to\left(\frac{L}{2\pi}\right)^3\int d^3\vec k\ldots$

D.O.S's: $\sum_if(\epsilon_i)=\sum_{\text{spin}}\sum_{\vec k}f(\epsilon(\vec k))\to\int_0^\infty g(\epsilon)f(\epsilon)d\epsilon$

BE: $g(\epsilon)=\alpha V\sqrt\epsilon/2$ • FD: $g(\epsilon)=\alpha V\sqrt\epsilon$ • $\alpha=4\pi(2m/h^2)^{3/2}$.

$N=\int_0^\infty\langle n\rangle(\epsilon)g(\epsilon)d\epsilon$ • $U=\int_0^\infty\epsilon\langle n\rangle(\epsilon)g(\epsilon)d\epsilon$

Fermi gas@T=0: • $\epsilon_F=\mu(T=0)$ • $k_F^3=3\pi^2n$

$n=2\alpha\epsilon_F^{3/2}/3$ • $\frac{U}{N}=3\epsilon_F/5$ • $PV=2U/3$

Black body • $\mu=0$ • $\sum_{\text{polariz}}\sum_{\vec k}\to2\left(\frac{L}{2\pi}\right)^3\int d^3\vec k\to\frac{V}{\pi^2c^3}\int_0^\infty\omega^2d\omega$

$\frac{U}{V}=\int_0^\infty u_\omega(\omega,T)d\omega=\int_0^\infty u(\epsilon,T)d\epsilon$ • $\epsilon=\hbar\omega$ • $u_\omega(\omega,T)=\frac{1}{\pi^2c^3}\frac{\hbar\omega^3}{e^{\beta\hbar\omega}-1}$

$u(\epsilon,T)=\frac{1}{\pi^2(\hbar c)^3}\frac{\epsilon^3}{e^{\beta\epsilon}-1}$ • $\hbar\omega_{\max}\approx2.82k_BT$ • $U=\frac{4\sigma_B}{c}VT^4$

$S=\frac{16\sigma_B}{3c}VT^3$ • $PV=U/3$ • Rad.E-flux: $e\sigma_BT^4$

Debye • $\sum_{\text{polariz}}^{\text{phon}}\sum_{\vec k}^{BZ}\to\frac{9N}{\omega_D^3}\int_0^{\omega_D}d\omega\,\omega^2\ldots$ • $\omega_D^3=\frac{k_B^3T_D^3}{\hbar^3}=\frac{6\pi^2Nv_s^3}{V}$

$U=\frac{9N}{\omega_D^3}\int_0^{\omega_D}\frac{\hbar\omega^3d\omega}{e^{\beta\hbar\omega}-1}=\frac{9N(k_BT)^4}{(k_BT_D)^3}\int_0^{T_D/T}\frac{x^3dx}{e^x-1}$ • $C_V=\frac{9Nk_BT^3}{T_D^3}\int_0^{T_D/T}\frac{x^3e^xdx}{(e^x-1)^2}$

$\overset{T<T_c}{\text{BEC}}$ • $\frac{N}{V}=\frac{2}{\sqrt\pi}\left[\frac{2\pi mk_BT_c}{h^2}\right]^{\frac32}\int_0^\infty\frac{\sqrt x\,dx}{e^x-1}$ • $\frac{N_{\text{norm}}}{N}=\left[\frac{T_c}{T}\right]^{\frac32}$ • $\frac{N_{\text{cond}}}{N}=1-\left[\frac{T_c}{T}\right]^{\frac32}$

$U=\gamma Nk_BT\left[\frac{T_c}{T}\right]^{\frac32}$ • $C_V=\frac{5\gamma Nk_B}{2}\left[\frac{T_c}{T}\right]^{\frac32}$ • $\gamma=\left[\int_0^\infty\frac{x^{3/2}dx}{e^x-1}\right]\Big/\left[\int_0^\infty\frac{x^{1/2}dx}{e^x-1}\right]\approx0.77$

Non-Rel. Quantum Mech.

$\langle f(j)\rangle = \sum_j f(j)\text{Prob}(j)$ • $\sigma_j^2 \equiv \langle (j-\langle j\rangle)^2\rangle = \langle j^2\rangle - \langle j\rangle^2 \hookleftarrow$

$\langle f(x)\rangle = \int f(x)\rho_{\text{prob}}(x)\,dx$ • $\sigma_j = \sqrt{\sigma_j\sigma_j} \leftarrow$ std.dev. •↑ var

$[AB,C] = A[B,C]+[A,C]B$ • $e^{i\lambda A}Qe^{-i\lambda A} = Q + i\lambda[A,Q] + \frac{(i\lambda)^2}{2!}[A,Q]^2 + \ldots$

Schrödinger Eqn.

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(\vec{r},t)\right)\Psi(\vec{r},t) \qquad (= H(\vec{r},\vec{p},t)\Psi).$$

Time-Indep. Schrödinger Eqn.

If $\partial_t V = 0$, using $\Psi(\vec{r},t) = \psi(\vec{r})\phi(t)$ ('sep of vars') leads to $H(\vec{r},\vec{p})\psi = E\psi$. General sol'n is $\Psi = \sum_n c_n\Psi_n(\vec{r},t)$ with $\Psi_n = \psi_n\phi_n = \psi_n(\vec{r})\exp\left(\frac{iE_nt}{-\hbar}\right)$ & $c_n = \langle\psi_n|\Psi_{t=0}\rangle = \int\psi_n^*(\vec{r})\Psi(\vec{r},t=0)d\vec{r} \mid E_n\in\mathbb{R},\ c_n\in\mathbb{C}$.

1. $\Psi_n \leftarrow$ *stationary states*; only phase evolves in t.
2. If V is symmetric $\Rightarrow$ each ψ_n (can be) even or odd.
3. Each $E_n > V_{\text{min}}$. (Else Ψ is not normalizeable.)
4. $E > V(-\infty)$ **or** $E > V(\infty) \Rightarrow$ scatter state(s).
5. $\psi_n \leftarrow$ an orthonormal set. (Completeness assumed.)
6. ψ continuous. $\Delta(\partial_x\psi) = \frac{2m}{\hbar^2}\lim_{\epsilon\to0}\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx$.
7. ψ_{n+1} has one more zero-crossing than ψ_n.
8. $i\hbar\partial_tQ(t) = [Q(t),H(t)] \quad\Rightarrow\quad \vec{p} = \frac{im}{\hbar}[H(t),\vec{r}]$

Interpretation, Requirements, Misc.

$\vec{p}\to\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}$ • $\vec{r}\to i\hbar\vec{\nabla}_{\vec{p}}$ • $[x,p_x] = i\hbar$ • $[f(x),p_x] = i\hbar\partial_xf$ • $\langle g|f\rangle^* = \langle f|g\rangle$

Eigen-funcs: $\widehat{x}:\quad x|\psi_y(x)\rangle = y|\psi_y(x)\rangle \Rightarrow |\psi_y(x)\rangle = \delta(x-y)$.

$\widehat{p}_x:\quad i\hbar\partial_x|\psi_{p_x}(x)\rangle = p_x|\psi_{p_x}(x)\rangle \Rightarrow |\psi_{p_x}(x)\rangle = e^{ip_xx/\hbar}/\sqrt{2\pi\hbar}$.

So: $\Phi(p_x,t)\equiv\langle\psi_{p_x}(x)|\Psi(x,t)\rangle$ and $\Psi(y,t)\equiv\langle\psi_{-p_x}(y)|\Phi(p_x,t)\rangle$

$\langle Q\rangle = \langle\Psi|\,[Q|\Psi\rangle] = [\langle\Psi|Q^\dagger]\,|\Psi\rangle = \sum_n q_n|c_n|^2$

$\langle Q(\vec{r},\vec{p})\rangle = \int\Psi^*Q\left(\vec{r},\frac{\hbar}{i}\vec{\nabla}_{\vec{r}}\right)\Psi d^3\vec{r} = \int\Phi^*Q\left(i\hbar\vec{\nabla}_{\vec{p}},\vec{p}\right)\Phi d^3\vec{p}$

Spectral decomp.: $Q = \sum_n q_nP_n$ if $Q|n\rangle = q_n|n\rangle$

$\rho_{\text{prob}}(x,t) = |\Psi(x,t)|^2 \iff \int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx = 1$

(but $\int_a^b|\Psi(x,t)|^2\,dx$ is time-dep. in general.)

$c_n = \langle\Psi_n|\Psi\rangle \overset{\text{prob.}}{\Longrightarrow} |c_n|^2$ • $c(z) = \langle\Psi_z|\Psi\rangle \overset{\text{prob.}}{\Longrightarrow} |c(z)|^2dz$

$E_n \overset{\text{prob.}}{\longleftrightarrow} |c_n|^2$ • $\sum_n|c_n(t)|^2 = 1$ • $\langle H\rangle = \sum_n|c_n|^2E_n$

Proj. op.: $P_n \equiv |n\rangle\langle n|/|\langle n|n\rangle|^2$ • $\sum_nP_n = 1$ • $\int P_z\,dz = 1$

Schwarz ineq.: $\langle a|a\rangle\langle b|b\rangle \geq |\langle a|b\rangle|^2 \geq ((\langle a|b\rangle - \langle a|b\rangle^*)/(2i))^2$

Uncertainty: $\sigma_A^2\sigma_B^2 \geq (\frac{1}{2i}\langle[A,B]\rangle)^2$ • $\sigma_x\sigma_{p_x}\geq\frac{\hbar}{2}$ • $\sigma_t\sigma_E\geq\frac{\hbar}{2}$

Correspondence principle → "classical" limit agrees.

Schrö. Pict.: $\langle Q(t)\rangle = \langle\Psi(x,t)|Q|\Psi(x,t)\rangle$

Heisen. Pict.: $\langle Q(t)\rangle = \langle\Psi(x,0)|e^{\frac{iHt}{\hbar}}Qe^{\frac{iHt}{-\hbar}}|\Psi(x,0)\rangle$

Feyn-Hllmn: $\partial_\lambda E_n = \langle\psi_n|\partial_\lambda H|\psi_n\rangle$ (no-degen or ψ is a proper lin. comb.)

Ehrenfest's Theorem / Virial Theorem

$\partial_t\langle Q\rangle = \frac{1}{i\hbar}\langle[Q,H]\rangle + \langle\partial_tQ\rangle$ • $\langle\ldots\rangle$'s obey classical laws.

$\frac{1}{m}\langle\vec{p}\rangle = \langle\vec{v}\rangle = \partial_t\langle\vec{r}\rangle$ • $\partial_t\langle\vec{p}\rangle = \langle-\vec{\nabla}V\rangle = \langle\vec{F}\rangle$ • $\partial_t\langle\vec{L}\rangle = \langle\vec{r}\times(-\vec{\nabla}V)\rangle$

If $V(\lambda\vec{r}_1,\lambda\vec{r}_2,\ldots) = \lambda^\nu V(\vec{r}_1,\vec{r}_2,\ldots)$ for N particles then $\langle n|T|n\rangle = \frac{\nu}{2}\langle n|V|n\rangle$ (& $E_n = \langle n|T|n\rangle + \langle n|V|n\rangle$).

$\partial_t\langle\vec{r}\cdot\vec{p}\rangle = 2\langle T\rangle - \langle\vec{r}\cdot\vec{\nabla}V\rangle$ • $\partial_t\langle\psi_n|Q|\psi_n\rangle = 0$ **if** $\partial_tQ = 0$

1D Scatter $(L_{-\infty}\to R_{+\infty})$ • $R+T=1$

$\psi_L = Ae^{ik_Lx} + Be^{-ik_Lx}$, $\psi_R = Fe^{ik_Rx} + \cancel{Ge^{-ik_Rx}}$ • $R\equiv\frac{|B|^2}{|A|^2}$ • $T\equiv\frac{v_R^{\text{group}}}{v_L^{\text{group}}}\frac{|F|^2}{|A|^2}$

1D Infinite Square Well

$V(x) = \begin{Bmatrix}0, & 0\leq x\leq L\\ \infty, & else\end{Bmatrix} \Longrightarrow \frac{d^2\psi}{dx^2} = -k^2\psi \,\Big|\, k\equiv\frac{\sqrt{2mE}}{\hbar}$.

$E_n = \frac{\hbar^2k_n^2}{2m} = \frac{\hbar^2n^2\pi^2}{2mL^2}$ • $\psi_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ • $n\in\mathbb{N}_{>0}$

$\langle n|x|n\rangle = \frac{L}{2}$ • $\langle n|x^2|n\rangle = \frac{L^2}{3} - \frac{L^2}{2\pi^2n^2}$ • $\langle n|p|n\rangle = 0$

$\langle n|p^2|n\rangle = \frac{\pi^2\hbar^2n^2}{L^2}$ • $\sigma_{x,n} = \sqrt{\frac{\pi^2n^2-6}{12}}\frac{L}{\pi n}$ • $\sigma_{p,n} = \hbar\frac{\pi n}{L}$

1D Harmonic Oscillator / 3D

$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$ • $H = \hbar\omega(a^\dagger a+\frac{1}{2}) = \hbar\omega(N+\frac{1}{2})$

$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right)$ • $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{ip}{m\omega}\right)$ • $[a,a^\dagger] = 1$

$[N,a^\dagger] = a^\dagger$ • $[a,N] = a$ • $N|n\rangle = n|n\rangle$ • $a|n\rangle = \sqrt{n}|n-1\rangle$

$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger+a\right)$ • $p = i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger-a\right)$ • $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$[H,a] = -\hbar\omega a$ • $[H,a^\dagger] = \hbar\omega a^\dagger$ • $[H,x] = \frac{\hbar}{im}p$ • $[H,p] = i\hbar m\omega^2x$

$E_n = \hbar\omega(n+\frac{1}{2})$ • $\psi_n = \frac{1}{\sqrt{2^nn!}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{\frac{m\omega x^2}{-2\hbar}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$ (hermite poly's)

$\psi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0$ • $n\in\mathbb{N}_{\geq0}$ • $H_n(x) = (-1)^ne^{x^2}\partial_x^n\left(e^{-x^2}\right)$

$\langle n|x|q\rangle = \sqrt{\hbar/2m\omega}(\sqrt{n}\delta_{q,n-1}+\sqrt{q}\delta_{n,q-1})$ • $\langle n|V|n\rangle = \frac{1}{2}E_n$

$\langle n|p|q\rangle = i\sqrt{m\omega\hbar/2}(\sqrt{n}\delta_{q,n-1}-\sqrt{q}\delta_{n,q-1})$ • $\langle n|T|n\rangle = \frac{1}{2}E_n$

$\langle n|x^2|q\rangle = (\hbar/2m\omega)\left((2q+1)\delta_{n,q} + \sqrt{n(q+1)}\delta_{n,q+2} + \sqrt{q(n+1)}\delta_{n,q-2}\right)$

$\langle n|p^2|q\rangle = (m\omega\hbar/2)\left((2q+1)\delta_{n,q} - \sqrt{n(q+1)}\delta_{n,q+2} - \sqrt{q(n+1)}\delta_{n,q-2}\right)$

$\psi = \psi_a(x)\psi_b(y)\psi_c(z)$ • $E_n = \hbar\omega(n+\frac{3}{2})$ • $d_n = (n+1)(n+2)/2$

1D Free Particle

$V(x) = 0$ • $\partial_x^2\psi = -k^2\psi$ • $p = \hbar k$ • $\lambda = \frac{2\pi}{|k|}$ • $\omega_k = \frac{\hbar k^2}{2m}$

$\left\{k\equiv\pm\frac{\sqrt{2mE}}{\hbar}\,\middle|\,\begin{matrix}k>0\text{ moves }\rightarrow\\ k<0\text{ moves }\leftarrow\end{matrix}\right\}$ • $\Psi_k\propto e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}$ (stationary states)

$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}dk \quad\leftarrow$ "Wave packet." General sol'n, with ~weights: $\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,t=0)e^{-ikx}dx$.

$v_{\text{phase}} = \frac{\omega}{|k|} = \frac{\hbar|k|}{2m} = \sqrt{\frac{E}{2m}}$ • $\overset{(\&\text{classical})}{v_{\text{group}}} = \frac{d\omega}{dk} = \frac{\hbar|k|}{m} = \sqrt{\frac{2E}{m}}$

1D Delta Well/Barrier

$V(x) = -\alpha\delta(x)$ • $\partial_x^2\psi = -\frac{2mE}{\hbar^2}\psi \equiv \overset{\text{bound}}{\kappa^2\psi} \equiv \overset{\text{scatter}}{-k^2\psi}$.

Bound $\to\kappa$ • $\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\exp\left(\frac{m\alpha|x|}{-\hbar^2}\right)$ • $E = -\frac{m\alpha^2}{2\hbar^2}$

Scatter $\to k$ • $R = \left[1+\left(\frac{2\hbar^2E}{m\alpha^2}\right)\right]^{-1}$ • $T = \left[1+\left(\frac{m\alpha^2}{2\hbar^2E}\right)\right]^{-1}$

$V(x) = \alpha\delta(x)$ • No bound • Same scatter as δ-well.

1D Double Delta Well

$V(x) = -\alpha(\delta(x+a)+\delta(x-a))$ • $\kappa\equiv\frac{\sqrt{-2mE}}{\hbar}$ • $k\equiv\frac{\sqrt{2mE}}{\hbar}$

Bound: $\left\{\begin{matrix}\text{Even: } e^{-2\kappa a} = \frac{\hbar^2\kappa}{m\alpha}-1\\ \text{Odd: } e^{-2\kappa a} = 1-\frac{\hbar^2\kappa}{m\alpha}\end{matrix}\,\middle|\,\begin{matrix}\text{Sol'n: always}\\ \text{Sol'n: sometimes}\end{matrix}\right\}$

Scatter: $T = \frac{1}{1+2z^2[(1+z^2)+(1-z^2)\cos(4ak)-2z\sin(4ak)]}$ (with $z = \alpha m/\hbar^2k$)

1D Finite Square Well / Barrier

$V(x) = \begin{Bmatrix}-V_0, & |x|\leq a\\ 0, & |x|>a\end{Bmatrix}$ • $\ell\equiv\frac{\sqrt{2m(E+V_0)}}{\hbar}$ • $\kappa\equiv\frac{\sqrt{-2mE}}{\hbar}$

Bound: $\left\{\begin{matrix}\text{Even: } \tan z = \sqrt{(z/z_0)^2-1}\\ \text{Odd: } \tan z = {}^{-1}/\sqrt{(z/z_0)^2-1}\end{matrix}\,\middle|\,\begin{matrix}z\equiv\ell a, \quad \&\\ z_0\equiv\frac{a}{\hbar}\sqrt{2mV_0}\end{matrix}\right\}$

Sol'ns: •Even: always, finite# •Odd: iff $V_0\geq\frac{\pi^2\hbar^2}{8m\alpha^2}$.

Scatter: $T^{-1} = 1+\frac{V_0^2}{4E(E+V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E+V_0)}\right)$

The 1D Finite Square Well, but $-V(x)$. No bound.

Scatter: $T_{E<V_0}^{-1} = 1+\frac{V_0^2}{4E(V_0-E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0-E)}\right)$

$T_{E=V_0}^{-1} = 1+\frac{2mE}{\hbar^2}a^2$ • $T_{E>V_0}^{-1} = 1+\frac{V_0^2}{4E(E-V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E-V_0)}\right)$

1D Sudden Step Up / Drop Down

$V(x) = \begin{Bmatrix}0, & x\leq0\\ V_0, & x>0\end{Bmatrix}$ • Reminder: $\Delta v_{\text{group}}|_{x=0}\neq0$.

$T_{E<V_0} = 0$ • $T_{E>V_0} = \sqrt{\frac{E-V_0}{E}}\frac{4E}{V_0^2}\left(\sqrt{E}-\sqrt{E-V_0}\right)^2$

Like Step Up, but $-V(x)$ • $T = 4(\sqrt{1+V_0/E})/(1+\sqrt{1+V_0/E})^2$

3D Spherical Symmetry & $\partial_tV = 0$

$\psi\equiv R(r)Y(\theta,\phi)$ • $u(r)\equiv rR(r)$ • $Y_l^{-m} \overset{m}{=} (-1)^m(Y_l^m)^*$

$\int_0^\infty|R|^2r^2dr = 1$ • $\int_0^{2\pi}\int_0^\pi(Y_l^m)^*Y_{l'}^{m'}\sin\theta d\theta d\phi = \delta_{l,l'}\delta_{m,m'}$

$Y_l^m(\theta,\phi) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos\theta)\,\Big|\,\epsilon = \begin{cases}1 & \text{if } m<0,\\ (-1)^m & \text{else.}\end{cases}$

$P_l^m(x) = (1-x^2)^{\frac{|m|}{2}}\partial_x^{|m|}P_l(x)$ • $P_l(x) = \partial_x^l(x^2-1)^l/l!2^l$ • $\partial_t\langle\vec{L}\rangle = 0$

$\frac{-\hbar^2}{2m}\partial_r^2u + \left[V+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$ • $[H,L^2] = [H,L_z] = 0$, $m\in-l,\ldots0,1,\ldots l$

3D Rigid $\circlearrowleft$ing Circle with N Masses

$N\circlearrowleft$ about $\vec{r}_\circ$ at $r=a$ • $H = T+\cancel{V} = \frac{1}{2}Mv^2$ • $|\vec{L}| = aMv$

$H = \frac{L^2}{2a^2M} \Rightarrow E_l = \frac{\hbar^2l(l+1)}{2a^2M}$ • $\psi_{lm} = Y_l^m$ • $m\in-l,..0,1,..l$, $l\in0,1,2,\ldots$

3D Inf. Spherical Well

$V(x) = \begin{Bmatrix}0, & r<a\\ \infty, & else\end{Bmatrix} \Longrightarrow \partial_r^2u = \left[\frac{l(l+1)}{r^2}-k^2\right]u \,\Big|\, k\equiv\frac{\sqrt{2mE}}{\hbar}$.

$u(r) = Arj_l(kr)+Brn_l(kr)$, $j_0 = \frac{\sin x}{x}$ • $n_0 = \frac{-\cos x}{x}$ • $j_l(x) = (-x)^l\left(\frac{1}{x}\partial_x\right)^l\frac{\sin x}{x}$, $n_l(x) = -(-x)^l\left(\frac{1}{x}\partial_x\right)^l\frac{\cos x}{x}$

$\left.\begin{matrix}j_l(x)\approx\frac{2^ll!x^l}{(2l+1)!}\\ n_l(x)\approx\frac{-(2l)!}{2^ll!x^{l+1}}\end{matrix}\right|x\ll1$ • $\left.\begin{matrix}j_l(x)\approx\frac{1}{x}\cos(x-\frac{\pi}{2}(l+1))\\ n_l(x)\approx\frac{1}{x}\sin(x-\frac{\pi}{2}(l+1))\end{matrix}\right|x\gg1$

$\partial_xj_l(x) = \frac{l}{x}j_l(x)-j_{l+1}(x)$ • $\partial_xn_l(x) = \frac{l}{x}n_l(x)-n_{l+1}(x)$

$E_{nl} = \frac{\hbar^2\beta_{nl}^2}{2ma^2}\,\Big|\,\begin{matrix}j_l(\beta_{nl}) = 0\ (n\text{ roots}/l)\\ E_{nl}\to(2l+1)\text{-degen.}\end{matrix}$ • $\psi_{nlm} = j_l(\beta_{nl}r/a)Y_l^m$

Hydrogen Atom – Bound States

$V(r) = \frac{-e^2}{4\pi\epsilon_\circ}\frac{1}{r}$ • $\kappa\equiv\frac{\sqrt{-2mE}}{\hbar}$ • $E_n = \frac{-m}{2\hbar^2n^2}\left(\frac{e^2}{4\pi\epsilon_\circ}\right)^2\,\Big|\,\begin{matrix}n\in\mathbb{N}_{>0},\\ n^2\text{-deg.}\end{matrix}$

$a_\circ = \frac{\hbar}{\alpha m_ec} = \frac{4\pi\epsilon_\circ\hbar^2}{m_ee^2} = \frac{1}{n\kappa_n}\approx0.529\text{Å}$ • $m\in-l,..0,1,..l$, $l\in0,1,..,(n-1)$

$\psi_{nlm} = \sqrt{\left[\frac{2}{na_\circ}\right]^3\frac{(n-l-1)!}{2n[(n+l)!]^3}}e^{\frac{-r}{na_\circ}}\left[\frac{2r}{na_\circ}\right]^lL_{(n+l)-(2l+1)}^{(2l+1)}\left(\frac{2r}{na_\circ}\right)Y_l^m(\theta,\phi)$

$L_{q-p}^p(x) = (-1)^p\partial_x^pL_q(x)$ • $L_q(x) = e^x\partial_x^q(e^{-x}x^q)$ • $\psi_{n00} = \frac{e^{-r/na_\circ}}{\sqrt{\pi(na_\circ)^3}}$

$|nlm\rangle\Rightarrow\begin{matrix}\langle V\rangle = 2E_n\\ \langle T\rangle = -E_n\end{matrix}$ • $\langle\frac{1}{r}\rangle = \frac{1}{a_\circ n^2}$ • $\langle\frac{1}{r^2}\rangle = \frac{1}{a_\circ^2n^3(l+1/2)}$ • $\langle p^2\rangle = \left(\frac{\hbar}{a_\circ n}\right)^2$

$\langle r\rangle = \frac{a_\circ}{2}[3n^2-l(l+1)]$ • $\frac{s+1}{n^2}\langle r^s\rangle - (2s+1)a_\circ\langle r^{s-1}\rangle + \frac{s}{4}[(2l+1)^2-s^2]a_\circ^2\langle r^{s-2}\rangle = 0$

$|100\rangle\Rightarrow\langle r^q\rangle = \frac{(q+2)!a_\circ^q}{2^{q+1}}$ • Most prob $r=a_\circ$ • $\langle z^2\rangle = a_\circ^2$, $\langle z\rangle = 0$ • $\langle z^q\rangle = \frac{(1+(-1)^q)(q+2)!a_\circ^q}{(q+1)2^{q+2}}$

Angular Momentum, Intrinsic Spin

$\vec{L} = \vec{r}\times\vec{p}$, $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ • $L^2 = \vec{L}\cdot\vec{L}$, $S^2 = \vec{S}\cdot\vec{S}$ • $[L_x,L_y] = i\hbar L_z$, $[S_x,S_y] = i\hbar S_z$ • $[L_y,L_z] = i\hbar L_x$, $[S_y,S_z] = i\hbar S_x$ • $[L_z,L_x] = i\hbar L_y$, $[S_z,S_x] = i\hbar S_y$

$[\vec{L},r^2] = 0$, $[\vec{L},p^2] = 0$ • $[\vec{L},\vec{S}] = 0$, $[L^2,\vec{S}] = 0$ • $[L^2,\vec{L}] = 0$, $[S^2,\vec{S}] = 0$ • $L_\pm\equiv L_x\pm iL_y$, $S_\pm\equiv S_x\pm iS_y$ • $[L_z,L_\pm] = \pm\hbar L_\pm$, $[S_z,S_\pm] = \pm\hbar S_\pm$

$[L_z,x] = i\hbar y$, $[L_z,p_x] = i\hbar p_y$ • $[L_z,y] = -i\hbar x$, $[L_z,p_y] = -i\hbar p_x$ • $[L_z,z] = 0$, $[L_z,p_z] = 0$ • $[\vec{L}\cdot\vec{S},\vec{L}] = i\hbar\vec{L}\times\vec{S}$, $[\vec{L}\cdot\vec{S},\vec{S}] = i\hbar\vec{S}\times\vec{L}$ • $[\vec{L}\cdot\vec{S},\vec{J}] = 0$, $[\vec{L}\cdot\vec{S},L^2] = 0$

$[\vec{L}\cdot\vec{S},S^2] = 0$, $[\vec{L}\cdot\vec{S},J^2] = 0$ • $L^2Y_l^m = \hbar^2l(l+1)Y_l^m$, $S^2|sm\rangle = \hbar^2s(s+1)|sm\rangle$ • $L_\pm Y_l^m = \hbar\sqrt{(l\mp m)(l\pm m+1)}Y_l^{m\pm1}$, $S_\pm|sm\rangle = \hbar\sqrt{(s\mp m)(s\pm m+1)}|sm\pm1\rangle$

$L_zY_l^m = \hbar mY_l^m$, $S_z|sm\rangle = \hbar m|sm\rangle$ • $L_z = -i\hbar\partial_\phi$, $(Y_l^m)^*xyY_l^m = 0$ • $s = 0,1/2,1,\ldots$, $m_s = -s,..,0,1,..,s$

$\vec{\sigma} = \left[\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&-i\\i&0\end{bmatrix},\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right]\to\begin{bmatrix}1\\\pm1\end{bmatrix},\begin{bmatrix}1\\\pm i\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}/\begin{bmatrix}0\\1\end{bmatrix}$ • $\sigma_j\sigma_k = \delta_{jk}+i\sum_n\epsilon_{jkn}^{(123)}\sigma_n$

$|j_1m_1\rangle|j_2m_2\rangle = \sum_jC_{m_1m_2m}^{j_1j_2j}|jm\rangle$ • $|jm\rangle = \sum_{m_1,m_2}C_{m_1m_2m}^{j_1j_2j}|j_1m_1\rangle|j_2m_2\rangle$

$J_z\to m\equiv m_1+m_2$ • $J^2\to j\in|j_1-j_2|,..(++1)..,(j_1+j_2)$

If $s_1 = 1/2$ then: (with $s = s_2\pm1/2$) • $|sm\rangle = A|\frac{1}{2}\frac{1}{2}\rangle|s_2(m-\frac{1}{2})\rangle + B|\frac{1}{2}\frac{-1}{2}\rangle|s_2(m+\frac{1}{2})\rangle$

$A = \sqrt{\frac{s_2\pm m+1/2}{2s_2+1}}$ • $B = \pm\sqrt{\frac{s_2\mp m+1/2}{2s_2+1}}$

Corrections to hydrogen

$E_n\sim\alpha^2$ • Fine struc $\sim\alpha^4$, $\gamma(v)\neq1$ & $\vec{S}\leftrightarrow\circlearrowleft$ • Lamb$\Delta\sim\alpha^5$ quant'zd $\vec{E}$ • Hyperfine$\sim\alpha^4m_e/m_p$, $\vec{\mu}_p\to\vec{B}\Rightarrow\vec{\mu}_e\leftrightarrow\vec{B}_p$

Fine struc: $H' = H'_{\text{rel}}+H'_{s-o} = \frac{-p^4}{8m^3c^2}+\frac{\hbar\alpha}{2m^2r^3}\vec{S}\cdot\vec{L}$ • $E_{nj} = \frac{E_0^0}{n^2}\left[1+\left(\frac{\alpha}{n}\right)^2\left(\frac{n}{j+1/2}-\frac{3}{4}\right)\right]$

$\vec{S}\cdot\vec{L} = (J^2-L^2-S^2)/2$ • $p^2\psi = 2m(E-V)\psi$ • Good #'s: n,l,s,j,m_j

Zeeman: $H' = -(\vec{\mu}_l+\vec{\mu}_s)\cdot\vec{B}_{\text{ext}} = \frac{e}{2m}(\vec{L}+2\vec{S})\cdot\vec{B}_{\text{ext}}$ •conflicts w/ fs

If: $B_{\text{ext}}\ll B_{\text{int}}\Rightarrow$fine struc domn8's$\Rightarrow E^1 = \langle nljm_j|H'|nljm_j\rangle$

use: $\vec{S}_{\text{t-avgd}} = \vec{J}(\vec{S}\cdot\vec{J})/J^2$ • $\vec{S}\cdot\vec{J} = (J^2-L^2+S^2)/2$ • $E_{\text{gs}}^1\sim\pm\mu_BB_{\text{ext}}$.

If: $B_{\text{ext}}\gg B_{\text{int}}\Rightarrow$zeeman domn8's$\Rightarrow$non-pert with$|nlm_lm_s\rangle$ align $||\widehat{z}\Rightarrow E^1 = \mu_BB_{\text{ext}}(m_l+2m_s)$.(note: can also pert the fs, use: $\langle\vec{S}\cdot\vec{L}\rangle = \cancel{\langle S_x\rangle\langle L_x\rangle}+\cancel{\langle S_y\rangle\langle L_y\rangle}+\langle S_z\rangle\langle L_z\rangle = \hbar^2m_lm_s$ if including H'_{fs}.)

If: $B_{\text{ext}}\sim B_{\text{int}}\Rightarrow$use full-blown degen pert (→ W eigenprob)

Linear Stark: $H' = \vec{E}_{\text{ext}}\cdot\vec{p}_e$. If $\vec{E}_{\text{ext}} = |\vec{E}|\widehat{z}$ then $H' = ezE_{\text{ext}} = eE_{\text{ext}}r\cos\theta$. Use degen pert. Most integrals→0.

Hyperfine: $H' = -\vec{\mu}_e\cdot\vec{B}_p\Rightarrow E^1 = A\langle[3(\vec{S}_p\cdot\widehat{r})(\vec{S}_e\cdot\widehat{r})-\vec{S}_p\cdot\vec{S}_e]/r^3\rangle + C|\psi(0)|^2\langle\vec{S}_p\cdot\vec{S}_e\rangle$ • $C = \mu_\circ g_pe^2/3m_pm_e = 8\pi A/3$ • $\vec{S}_p\cdot\vec{S}_e = (S^2-S_e^2-S_p^2)/2$

If: $l = 0\Rightarrow$ A term→0 $\Rightarrow E^1 = (S^2-3\hbar^2/2)2g_p\hbar^2/3m_pm_e^2c^2a_\circ^4$.

Breaks spin degen of the gs: $S^2 = 0$ for singlet, $2\hbar^2$ triplet.

Stationary e^- in Magnetic Field

$\vec{\mu} = \gamma\vec{S}$, $\vec{\tau} = \vec{\mu}\times\vec{B}$ • $H = \cancel{T}-\vec{\mu}\cdot\vec{B}$ if $\vec{B} = B_\circ\widehat{z}$ • $E_\pm = \mp\gamma B_\circ\hbar/2$, $\langle S_z\rangle = \frac{\hbar}{2}\cos\alpha$ • $\langle S_x\rangle\propto\cos\omega$, $\langle S_y\rangle\propto\sin\omega$ • $\omega = \gamma B_\circ$ Larmor

e^- Stern-Gerlach: $\vec{B}(x,y,z)\to\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})\neq0$

$\vec{B} = \cancel{-\alpha x\widehat{i}}+(B_\circ+\alpha z)\widehat{k}$, $H(t) = \begin{Bmatrix}\vec{\mu}\cdot\vec{B} & t\in[0,T]\\ 0 & \text{else}\end{Bmatrix}$ • $E_\pm = \mp\gamma(B_\circ+\alpha z)\hbar/2$ • $p_z = \alpha\gamma\hbar/2$ • $q\equiv i\gamma B_\circ/2$

$\chi(t) = \begin{bmatrix}a\chi_++b\chi_- & t<0\\ ae^{(qT+ip_zz)}(\chi_+)+be^{-(qT+ip_zz)}(\chi_-) & t>T\end{bmatrix}$

T.I. Perturbation Theory: $H\equiv H^0+\lambda H'$

$H^0|\psi_n^0\rangle = E_n^0|\psi_n^0\rangle$, $E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle$ • $|\psi_n^1\rangle = \sum_{m\neq n}\frac{\langle\psi_m^0|H'|\psi_n^0\rangle}{E_n^0-E_m^0}|\psi_m^0\rangle$ • $E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0-E_m^0}$

If: $H^0|\psi_a^0\rangle = E^0|\psi_a^0\rangle$ and $H^0|\psi_b^0\rangle = E^0|\psi_b^0\rangle$ and maybe more... so $H^0|\psi_{☺}^0\rangle = E^0|\psi_{☺}^0\rangle$ w/ $|\psi_{☺}^0\rangle = \alpha|\psi_a^0\rangle + \beta|\psi_b^0\rangle+\ldots$ • We need to determine the α,β that will split the degen. ψ's into distinct e-val's cleanly when H' is applied. There are two ways. [1] find $A = A^\dagger$ s.t. $[A,H^0] = [A,H'] = 0$ which are NOT degen. for the ψ's in E^0s sub-space $(\psi_a^0,\psi_b^0,\ldots)$ and use non-degen. [2] solve **W** eigenprob for α,β and E^1(s). You hopefully get N E's and req'd α,β.

$\begin{pmatrix}W_{aa}&W_{ab}\\W_{ba}&W_{bb}\end{pmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = E^1\begin{bmatrix}\alpha\\\beta\end{bmatrix}\,\Big|\,\begin{matrix}W_{ij} = \langle\psi_i^0|H'|\psi_j^0\rangle\\ |\psi_{☺}^0\rangle = \alpha|\psi_a^0\rangle+\beta|\psi_b^0\rangle+\ldots\end{matrix} \Rightarrow$ Solve $N\times N$: $\det(\mathbf{W}-\lambda) = 0$

Variational Principle

$E_{gs}\leq\langle\psi|H|\psi\rangle$ for any normalized $|\psi\rangle$. Simply minimize!

V∝x: $\partial_x^2\psi = \alpha^3x\psi\to\psi = C\,Ai(x)+D\,Bi(x)$ (Airy's)

$Ai(x\gg0)\sim e^{-\frac{2}{3}x^{3/2}}/(2\sqrt{\pi}x^{1/4})$, $Bi(x\gg0)\sim e^{+\frac{2}{3}x^{3/2}}/(\sqrt{\pi}x^{1/4})$ • $Ai(x\ll0)\sim\sin\left[\frac{2}{3}(-x)^{3/2}+\frac{\pi}{4}\right]/(\sqrt{\pi}(-x)^{1/4})$, $Bi(x\ll0)\sim\cos\left[\frac{2}{3}(-x)^{3/2}+\frac{\pi}{4}\right]/(\sqrt{\pi}(-x)^{1/4})$

T.D. Perturbation Theory: $H\equiv H^0+\lambda H'(t_{>0})$

$H^0\psi_n^0 = E_n^0\psi_n^0$, $H'_{mn}\equiv\langle\psi_m^0|H'|\psi_n^0\rangle$ • $\Psi(t) = \sum_nc_n(t)\psi_ne^{-iE_nt/\hbar}$ • $\Delta_{mn}\equiv E_m-E_n$, $\partial_tc_m = \frac{1}{i\hbar}\sum_nc_nH'_{mn}e^{-i\Delta_{mn}t/\hbar}$ (exact)

(F.O.) $c_n(t)\approx1+\frac{1}{i\hbar}\int_0^tH'_{nn}(t')dt'$ • $c_{m\neq n}(t)\approx\frac{1}{i\hbar}\int_0^tH'_{mn}(t')e^{-i\Delta_{mn}t'/\hbar}dt'$

$\partial_tH' = 0$: $P_{n\to m} = \frac{\sin^2(\Delta_{nm}t/2\hbar)}{\Delta_{nm}^2/(4|H'_{mn}|^2)}$ • $H' = V\cos\omega t$: $P_{n\to m} = \frac{\sin^2((\Delta_{nm}\pm\omega\hbar)t/2\hbar)}{(\Delta_{nm}\pm\omega\hbar)^2/(|V_{mn}|^2)}$ iff $\Delta_{nm} = \pm\hbar\omega$

Spon. emission rate: $A = \frac{\omega_0^3|\mathcal{P}|^2}{3\pi\epsilon_\circ\hbar c^3}$ • $\mathcal{P} = q\langle\psi_b|\vec{r}|\psi_a\rangle$ • Life-time: $\tau = \frac{1}{A_1+A_2+\ldots}$

WKB(J) – (Classical endpoints a,b; $L\to R$)

Assume $\psi = A(x)e^{i\phi(x)}$ & $\partial_x^2[A]/A\ll((\partial_x\phi)^2$ & $p^2/\hbar^2)$ • $\psi(x)\approx C\exp\left(\frac{\pm i}{\hbar}\int_a^bp(x')dx'\right)/\sqrt{p(x)}$, $p(x) = \sqrt{2m(E-V_\circ)}$ "classical" p.

Tunneling $a\to b$: $T = |F|^2/|A|^2\approx e^{-2\gamma}$ • $\gamma\hbar\equiv\int_a^b\sqrt{|p(x)|^2}dx$

2 |walls: $\int_a^bp(x)dx = n\pi\hbar$, 1 |wall @a: $\int_a^bp(x)dx = (n-\frac{1}{4})\pi\hbar$ • no |walls: $\int_a^bp(x)dx = (n-\frac{1}{2})\pi\hbar$ w/ $n = 1,2,..$ • Best when: $n\gg1$ (semi-classical regime)

$\langle T\rangle = \frac{1}{2}\partial_n[E_n](n-\frac{1}{2})$ when there are no |walls.

Scattering: Partial wave, Born

$D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2$, $\sigma = \int D(\theta)\,d\Omega$ • $\psi(r,\theta)\approx A[e^{ikz}+f(\theta)\frac{e^{ikr}}{r}]$, $h_0^{(1)} = \frac{e^{ix}}{ix}$ • $h_l^{(1)}\equiv j_l(x)+in_l(x)$ • $u(r) = Arj_l(kr)+Brn_l(kr)$ Use u near origin, match with expansion.

$e^{ikz} = \sum_{i=0}^\infty i^l(2l+1)j_l(kr)P_l(\cos\theta)$ • $\sigma = 4\pi\sum_{i=0}^\infty(2l+1)|a_l|^2$

$\psi(r,\theta) = A\sum_{i=0}^\infty i^l(2l+1)[j_l(kr)+ika_lh_l^{(1)}(kr)]P_l(\cos\theta)$ ←impose b.c.'s.

1. $V(\vec{r}_0)$ is localized about $\vec{r}_0$ 2. incoming wave ~ not altered.

$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|}V(\vec{r}_0)\psi(\vec{r}_0)d^3\vec{r}_0$ s.t. $(\nabla^2+k^2)\psi_0 = 0$

Born$\to f(\theta,\phi)\approx\frac{-m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}'}V(\vec{r}')d^3\vec{r}'$ s.t. $\vec{k} = k\widehat{r},\vec{k}' = k\widehat{z}$

Low-E: $f(\theta,\phi)\approx\frac{-m}{2\pi\hbar^2}\int V(\vec{r}')d^3\vec{r}'$ • Spherical sym: $\kappa = 2k\sin(\theta/2)$, $f(\theta)\approx\frac{-2m}{\kappa\hbar^2}\int rV(r)\sin(\kappa r)dr$

1. $V(\vec{r}_0)$ is localized about $\vec{r}_0$ 2. incoming wave ~ not altered.

Electrodynamics

	General	In matter	Aux. & Pot.s
Gauss'	$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_\circ$	$\vec{\nabla}\cdot\vec{D} = \rho_f$	$\vec{D} = \epsilon_\circ\vec{E}+\vec{P}$
Faraday's	$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$	$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$	$\vec{H} = \frac{1}{\mu_\circ}\vec{B}-\vec{M}$
No monopoles	$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{E} = -\vec{\nabla}V-\partial_t\vec{A}$
Ampere's	$\vec{\nabla}\times\vec{B} = \mu_\circ(\vec{J}+\epsilon_\circ\partial_t\vec{E})$	$\vec{\nabla}\times\vec{H} = \vec{J}_f+\partial_t\vec{D}$	$\vec{B} = \vec{\nabla}\times\vec{A}$

Linear media: $\vec{P} = \epsilon_\circ\chi_e\vec{E}$ so $\vec{D} = \epsilon\vec{E}$ & $\vec{M} = \chi_m\vec{H}$ so $\vec{H} = \frac{1}{\mu}\vec{B}$.

Enrg: $U = \frac{1}{2}\int(\epsilon_\circ E^2+\frac{1}{\mu_\circ}B^2)dV$, Momntm: $\vec{P} = \epsilon_\circ\int(\vec{E}\times\vec{B})dV$ • Poynting: $\vec{S} = \frac{1}{\mu_\circ}(\vec{E}\times\vec{B})$, Larmor: $P = \frac{\mu_\circ}{6\pi c}q^2a^2$ • $C\equiv Q/V$, $W = CV^2/2$

$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B})$, $\frac{-\sigma}{\epsilon_\circ} = \widehat{n}\cdot\vec{\nabla}V\Big|_{@\text{surf}}$ • $W = Q[V(\vec{r})-V(\mathcal{O})]$, $W = \frac{1}{2}\int V\,dq$ • $\vec{F} = \widehat{n}\sigma^2/2\epsilon_\circ$, $P = E^2\epsilon_\circ/2$ surf. press. •

Spherical image: $|r_q| = a\to q' = -qR/a$ & $|r_{q'}| = R^2/a$ • $V(\vec{r}) = k((q/r)+(q'/r'))$

$-\nabla^2V = \cancel{\rho/\epsilon_\circ}$ (Laplace) solve via sep.vars: $V\equiv X(x)Y(y)Z(z)\to\frac{1}{X}\partial_x^2X = C\to$B.c.'s!

$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\sum_i^N\frac{q_i\widehat{r}}{r_i^2}$, $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int_{\mathbb{V}}\frac{1}{r^2}\widehat{r}dq$ $\Big|$ $dq = \{\lambda dl,\sigma da,\rho dV\}$, $\vec{r} = \vec{r}-\vec{r}'$, q at $\vec{r}'$. • $\oint\vec{E}\cdot d\vec{a} = \frac{Q_{\text{enc}}}{\epsilon_\circ}$, $\vec{E} = -\vec{\nabla}V$, $\vec{E} = \frac{\sigma\widehat{n}}{2\epsilon_\circ}$ (∞ plane) • $V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}}\vec{E}\cdot d\vec{l}$, $V(\vec{r}) = \frac{1}{4\pi\epsilon_\circ}\int_{\mathbb{V}}\frac{1}{r}dq$

Geometry – $4\pi\epsilon_\circ V(\vec{r})$ • $4\pi\epsilon_\circ\vec{E}(\vec{r})$

1D wire. $\|\|\widehat{x}$. Len: L ⊥ above cent.	$\lambda\ln\left(\frac{L}{z}+\sqrt{1+(L/z)^2}\right)$	$\frac{(\lambda L)}{z\sqrt{z^2+L^2/4}}\widehat{z}$
2D ring. $\widehat{x}-\widehat{y}$. ⊥ above cent.	$\frac{(\lambda2\pi R)}{(R^2+z^2)^{1/2}}$	$\frac{z(\lambda2\pi R)}{(R^2+z^2)^{3/2}}\widehat{z}$
2D disk. $\widehat{x}-\widehat{y}$. ⊥ above cent.	$\frac{q}{R^2}\left(\sqrt{z^2+R^2}-\|z\|\right)$	$2\frac{(\sigma\pi R^2)}{R^2}\left(1+\frac{-\|z\|}{\sqrt{z^2+R^2}}\right)\widehat{z}$
3D sph. shell. At origin.	$\frac{q}{r}$ (out), $\frac{q}{R}$ (in)	$\frac{(\sigma4\pi R^2)}{r^2}\widehat{r}$ | $r>R$, 0else

Magnetic Dipoles ($\vec{\mu}$ – proton, electron)

$\vec{\mu}_p = \frac{g_pe}{2m_p}\vec{S}_p$ • $\vec{\mu}_e = \frac{-e}{m_e}\vec{S}_e$ • $\vec{B}_\mu = \frac{\mu_0}{4\pi r^3}[3(\vec{\mu}\cdot\widehat{r})\widehat{r}-\vec{\mu}]+\frac{2\mu_0}{3}\vec{\mu}\delta^3(\vec{r})$

Electric Dipoles ($\pm|q|$ sep. by $\vec{d}\Rightarrow\vec{p}_e = -q\vec{d}$)

$\vec{p}_e = \int\vec{r}'\rho(\vec{r}')d^3\vec{r}'$ • $\frac{V}{k} = \frac{q}{r_+}-\frac{q}{r_-}\approx\frac{\vec{p}\cdot\widehat{r}}{r^2}$ • $\frac{\vec{E}}{k} = \frac{1}{r^3}[3(\vec{p}\cdot\widehat{r})\widehat{r}-\vec{p}]-\frac{4\pi}{3}\vec{p}\delta^3(\vec{r})$