Pictures (Operators are T.I. before evolution!)

Schr: $|\psi;t\rangle = U(t)|\psi\rangle$ $\quad A(t)=A$ $\quad i\hbar\frac{\partial}{\partial t}U(t) = H(t)U(t)$ $\quad |\psi;t\rangle = \sum_E e^{\frac{-iE(t-t_0)}{\hbar}}|E\rangle\langle E|\psi\rangle$ $\Big|$ $H(t)=0$

$$\langle A\rangle = \langle\psi;t|A|\psi;t\rangle = \sum_{E,E'}\psi^*(E)\psi(E')\langle E|A|E'\rangle e^{\frac{-i(E'-E)(t-t_0)}{\hbar}} \quad \Big| \quad H(t)=0 \qquad i\hbar\partial_t\Psi(x,t) = \left(\frac{-\hbar^2}{2m}\nabla^2 + V(x)\right)\Psi(x,t)$$

Heis: $|\psi;t\rangle = |\psi\rangle$ $\quad A(t) = U^\dagger(t)AU(t)$ $\quad \frac{\partial}{\partial t}A(t) = \frac{1}{i\hbar}[A(t),H(t)]$

Intr: $|\psi;t\rangle_I = e^{\frac{-iH_0 t}{\hbar}}|\psi;t\rangle_S$ $\quad A_I(t) = e^{\frac{iH_0t}{\hbar}}A_S(t)e^{\frac{-iH_0t}{\hbar}}$ $\quad i\hbar\frac{d}{dt}|\psi;t\rangle_I = V_I(t)|\psi;t\rangle_I$

$$|\alpha;t\rangle_I = \sum_n C_n(t)|n\rangle_S: \quad i\hbar\begin{pmatrix}\dot C_0\\ \dot C_1\\ \dot C_2\\ \vdots\\ \dot C_d\end{pmatrix} = \begin{pmatrix} V_{00} & V_{01}e^{i\omega_{01}t} & V_{02}e^{i\omega_{02}t} & \dots & V_{0d}e^{i\omega_{0d}t}\\ V_{10}e^{i\omega_{10}t} & V_{11} & V_{12}e^{i\omega_{12}t} & \dots & V_{1d}e^{i\omega_{1d}t}\\ V_{20}e^{i\omega_{20}t} & V_{21}e^{i\omega_{21}t} & V_{22} & \dots & V_{2d}e^{i\omega_{2d}t}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ V_{d0}e^{i\omega_{d0}t} & V_{d1}e^{i\omega_{d1}t} & V_{d2}e^{i\omega_{d2}t} & \dots & V_{dd}\end{pmatrix}\begin{pmatrix}C_0\\ C_1\\ C_2\\ \vdots\\ C_d\end{pmatrix} \quad \Big| \quad \begin{matrix} V_{nm} \equiv_S \langle n|V_I(t)|m\rangle_S\\ \omega_{nm} \equiv \frac{E_n-E_m}{\hbar}\end{matrix}$$

Born measurement rule

$$|\psi\rangle \xrightarrow[\text{obtain } a]{\text{meas. } A} \frac{P_a|\psi\rangle}{\sqrt{\langle\psi|P_a|\psi\rangle}} \quad \Big| \quad prob(a) = \langle\psi|P_a|\psi\rangle \quad \Big| \quad P_a \equiv \sum_i |a_i\rangle\langle a_i| \quad \Big| \quad \{|a_i\rangle\} \text{ degen. basis spans sub-Hilbert space}$$

Hilbert Spaces

$$\mathbb{H}_1\otimes\mathbb{H}_2\otimes\cdots\otimes\mathbb{H}_n = \mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\cdots\otimes\mathbb{C}^{d_n} = \mathbb{C}^{d_1\times d_2\times\cdots\times d_n} \Rightarrow \text{ dim of joint space } = d_1\times d_2\times\cdots d_n \quad \Big| \quad d_i \text{ is dim of } \mathbb{H}_i$$

Special Operator Types

Proj: $A^2 = A^\dagger = A$ $\quad$ Idnt: $\mathbb{I} = \sum_a |a\rangle\langle a| = \int d\eta\,|\eta\rangle\langle\eta|$ $\quad$ Cmptbl: $[A,B]=0$ $\quad$ Untry: $U^\dagger U = UU^\dagger = \mathbb{I}$ $\quad$ Obsrv: *herm.*

Herm: $A^\dagger = A \Rightarrow$ e-vals $\in\mathbb{R}$ $\quad$ Anti-H: $A^\dagger = -A \Rightarrow$ e-vals $\in\mathbb{C}$ $\quad$ Dnst: *herm*, $Tr(\rho) = \sum_i\langle i|\rho|i\rangle = 1$, e-vals $\in[0,1]\in\mathbb{R}$

Anti-Untry (time rev.): $\Theta = U\mathcal{K}\Big|$ $U_{\text{unitary}}, \mathcal{K}_{\text{cmplx-conj oper in a basis}}$ $\quad \langle\beta|A_{obs}|\alpha\rangle = \langle\tilde\alpha|\Theta A_{obs}\Theta^{-1}|\tilde\beta\rangle$ $\quad \Theta J\Theta^{-1} = -J$ $\quad \Theta X\Theta^{-1} = X$ $\quad \Theta P\Theta^{-1} = -P$ $\quad |\tilde\alpha\rangle = \Theta|\alpha\rangle$ $\quad |\tilde\beta\rangle = \Theta|\beta\rangle$ $\quad \Rightarrow \langle\tilde\alpha|\tilde\beta\rangle = (\langle\alpha|\beta\rangle)^*$

$\Theta^2|j,m\rangle = \pm|j,m\rangle \rightarrow$ + for j half-integer, - for j integer.

Special Operators

$$U(t,t_0) = e^{\frac{-i}{\hbar}\int_{t_0}^t dt' H(t')} \quad \Big| \quad [H(t_1),H(t_2)] = 0 \qquad \text{Trans: } T(dx) = \left\{\begin{matrix}\int dx\,|x+dx\rangle\langle x|\\ e^{\frac{-idxP}{\hbar}}\\ T^{-1}(-dx) = T^\dagger(-dx)\end{matrix}\right\} \quad \Big| \quad T(dx)|x\rangle = |x+dx\rangle$$

Parity: $\Pi^2 = \mathbb{I}$ $\quad \Pi^\dagger X\Pi = -X$, $\Pi^\dagger P\Pi = -P$ $\quad \Pi^\dagger J\Pi = J$, $\Pi^{-1} = \Pi^\dagger = \Pi$ $\quad \Pi|x\rangle = |-x\rangle$ $\quad \Pi|n,l,m\rangle = (-1)^l|n,l,m\rangle$, if $[H,\Pi]=0\Rightarrow$ even, odd e-states $\quad \Pi|\alpha\rangle = \pm|\alpha\rangle$ $\Big|$ { even: +, odd: -}

Ensembles, Mixed, Pure States

Ens: $\mathcal{E} = \{p_i, |\psi_i\rangle\}$ $\quad \langle A\rangle_{\mathcal{E}} = \sum_i p_i\langle\psi_i|A|\psi_i\rangle = Tr(A\rho)$ $\Big|$ $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| = \rho^\dagger$ $\quad Tr(\rho^2)\in[0,1]\rightarrow 1$ pure

Ehrenfest's theorem, Useful Relations

$$[X_i, F(P)] = i\hbar\frac{\partial F}{\partial P_i} \qquad [P_i, G(X)] = -i\hbar\frac{\partial G}{\partial X_i} \qquad m\frac{\partial^2}{\partial t^2}\langle X\rangle = -\langle\nabla V(X)\rangle$$

Angular Momentum

$[J_i,J_j] = \epsilon_{ijk}i\hbar J_k$ for J,S,L $(SU(2))$ $\quad L^2 = X^2P^2 - (X\cdot P)^2 + i\hbar X\cdot P$ $\quad S_i = \frac{\hbar}{2}\sigma_i$ $\quad \sigma_{x,y,z}, \sigma^\dagger_{x,y,z} = \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}0&-i\\i&0\end{bmatrix}, \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $\quad \{\sigma_i,\sigma_j\} = 2\delta_{i,j}$ $\quad \sigma_i^2 = \mathbb{I}$ $\quad L_i = \epsilon_{ijk}X_jP_k$ $\quad [G_i,G_j] = \epsilon_{ijk}G_k\ (SO(3))$

$[J^2,J_i] = 0$ $\quad J_\pm = J_x \pm iJ_y$ $\quad [J_+,J_-] = 2\hbar J_z$ $\quad [J_z,J_\pm] = \pm\hbar J_\pm$ $\quad J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$ $\quad J_z|j,m\rangle = \hbar m|j,m\rangle$ $\quad j\in\{0,\frac{1}{2},1,\frac{3}{2},\dots\}$ (int or half int) $\quad m = \{-j,-j+1,\dots,j-1,j\}$ $\quad$ Rot's: $\mathcal{D}(R(\theta,\vec n)) = e^{\frac{1}{i\hbar}\theta\vec n\cdot\vec J}$

Spherical Harmonics

$$\langle X|n,l,m\rangle \equiv R_{n,l}(r)Y_l^m(\theta,\phi) \quad \Big| \quad \begin{matrix}[H,L^2]\\ [H,L_i]\end{matrix} = 0 \quad Y_l^m(\theta,\phi) = e^{im\phi}\tilde Y_l(\theta) \quad \left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right) + V(r) - E_{n,l}\right]R_{n,l}(r) = 0$$

Addition of Ang. Mom.

$|j_1,j_2,j,m\rangle = \sum_{m_1,m_2}|j_1,j_2,m_1,m_2\rangle\langle j_1,j_2,m_1,m_2|j_1,j_2,j,m\rangle$ $\quad m_1\in\{-j_1,-j_1+1,\dots,j_1\}$ $\quad m_2\in\{-j_2,-j_2+1,\dots,j_2\}$ $\quad m_1+m_2\le j$ top right $\quad m_1+m_2\ge -j$ bot left

$J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm1\rangle$ $\quad |j_1-j_2|\le j\le j_1+j_2$ (integer sep.) $\quad \sum_{m_1,m_2}|\langle j_1,j_2,m_1,m_2|j_1,j_2,j,m\rangle|^2 = 1$

(0) Draw out grid with space for m and CGC. Compute $m = m_1+m_2$ for each site. Write in each CGC as computed. **Always from center!**

(1) Top Right, point A: $\langle j_1,j_2,m_1,m_2|j_1,j_2,j,m_1+m_2\rangle \equiv \langle m_1,m_2|j,m_1+m_2\rangle = 1$.

(2) Below A, dwnwrd: $\langle m_1,m_2|j,m-1\rangle^{cntr}_{new}$ J_+: (with $m = m_1+m_2+1$) $= \sqrt{\frac{(j_2-m_2)(j_2+m_2+1)}{(j+m)(j-m+1)}}\langle m_1,m_2+1|j,m\rangle_{old}$

(3) At A, go L: $\langle m_1-1,m_2|j,m\rangle_{new}$ J_-: (with $m = m_1+m_2-1$) $= \sqrt{\frac{(j-m)(j+m+1)}{(j_1+m_1)(j_1-m_1+1)}}\langle m_1,m_2|j,m+1\rangle^{cntr}_{old} - \sqrt{\frac{(j_2+m_2)(j_2-m_2+1)}{(j_1+m_1)(j_1-m_1+1)}}\langle m_1,m_2-1|j,m\rangle_{old}$

(4) Decrement down and left.

(5) L of A, go U: $\langle m_1,m_2+1|j,m\rangle_{new}$ J_+: (with $m = m_1+m_2+1$) $= \sqrt{\frac{(j+m)(j-m+1)}{(j_2-m_2)(j_2+m_2+1)}}\langle m_1,m_2|j,m-1\rangle^{cntr}_{old} - \sqrt{\frac{(j_1-m_1)(j_1+m_1+1)}{(j_2-m_2)(j_2+m_2+1)}}\langle m_1+1,m_2|j,m\rangle_{old}$

Uncertainty Relations

$$\left\langle(\Delta A)^2\right\rangle_{|\psi\rangle}\left\langle(\Delta B)^2\right\rangle_{|\psi\rangle} \ge \frac{1}{4}\left|\langle[A,B]\rangle_{|\psi\rangle}\right|^2 \Bigg| \left\{\begin{matrix}\Delta A = A - \langle A\rangle_{|\psi\rangle}\\ \left\langle(\Delta A)^2\right\rangle_{|\psi\rangle} = \left\langle A^2\right\rangle_{|\psi\rangle} - \langle A\rangle^2_{|\psi\rangle}\end{matrix}\right\} \Rightarrow \left\langle(\Delta X)^2\right\rangle_{|\psi\rangle}\left\langle(\Delta P)^2\right\rangle_{|\psi\rangle} \ge \frac{\hbar^2}{4} \text{ (Heisen.)}$$

State Vectors, Wavefunctions, Continua

State: $|\psi\rangle = \int dx'\,|x'\rangle\langle x'|\psi\rangle$ (indep of basis) $\quad$ Wavefunc: $\psi(x) = \langle x'|\psi\rangle$ (dep on basis) $\quad \langle x|x'\rangle = \delta(x-x')$

Commutation Relations and Representations

$[X_i, P_j] = i\hbar\delta_{i,j}$ $\quad [X_i, X_j] = [P_i, P_j] = 0$ $\quad P = \frac{\hbar}{i}\frac{d}{dX}$ $\quad \langle x| P |x'\rangle = \delta(x - x')\frac{\hbar}{i}\frac{d}{dX}$ $\quad \langle x| P |p\rangle = p\langle x| p\rangle$ $\quad \langle x| p\rangle \propto e^{\pm ipx}$

SHO's

$\hat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$ $\quad \hat{N} \equiv a^\dagger a$ $\quad \left[a, a^\dagger\right] = 1$ $\quad \hat{X} = \hat{X}^\dagger = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$ $\quad \hat{P} = \hat{P}^\dagger = i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right)$ $\quad \left[\hat{H}, \hat{P}\right] = i\hbar m\omega^2\hat{X}$

$\left[\hat{H}, \hat{X}\right] = \frac{\hbar}{im}\hat{P}$ $\quad a(t) = e^{-i\omega t}a$ $\quad a^\dagger(t) = e^{i\omega t}a^\dagger$ $\quad \hat{X}(t) = \cos(\omega t)\hat{X} + \sin(\omega t)\frac{\hat{P}}{m\omega}$ $\quad \hat{P}(t) = \cos(\omega t)\hat{P} - m\omega\sin(\omega t)\hat{X}$

No-Cloning

$M|\psi\rangle \neq |\psi\rangle \otimes |\psi\rangle$ | $\{\forall |\psi\rangle \in \mathbb{H}, \quad \forall M\}$ by linearity of all M: $\quad M(|0\rangle + |1\rangle) = |0\rangle|0\rangle + |1\rangle|1\rangle \neq |0\rangle|1\rangle \otimes |0\rangle|1\rangle$

Noether's Theorem

if $G = G^\dagger$ and $[G, H] = 0$ then G is conserved, and $e^{\frac{id\alpha G}{\hbar}} H e^{\frac{-id\alpha G}{\hbar}} = H$ (invariance under transform)

Tight-Binding, Bloch's Theorem

Tght-Bndg: $\langle n| H |n\rangle = E_0, \ \langle n \pm 1| H |n\rangle = -\Delta$ | $\forall\{n$ spatial peaks $\}$

$H|\theta\rangle = (E_0 - 2\Delta\cos\theta)|\theta\rangle$

$|\theta\rangle = \sum_n e^{in\theta}|n\rangle$ | $\theta \in [-\pi, \pi)$

$\langle x| \theta\rangle = e^{ikx}U_k(x)$ | $ka = \theta, \ U_k(x + a) = U_k(x)$

Kramer's Degeneracy

For odd nmbr of e^- in t.i.electro field, *every enrg-lvl is degen.* $\quad V(t) = V = -e\phi$ | $\vec{\mathcal{E}} = -\nabla\phi$ $\quad$ Proof: use time-rev $\Theta^2|n\rangle = |n\rangle$!

WKBJ

$\Psi(x, t) = \sqrt{\rho(x, t)}e^{iS(x,t)/\hbar}$ $\quad \hbar\left|\nabla^2 S\right| \ll |\nabla S|^2$ $\quad \Psi(x, t) = \frac{const}{(E - V(x))^{1/4}}exp\left(\frac{\pm}{\hbar}\int_{-\infty}^{x}\sqrt{2m(V(x') - E)}dx' \ - i\frac{Et}{\hbar}\right)$

$\lambda = \frac{\hbar}{\sqrt{2m(E - V(x))}} \ll \frac{2(E - V(x))}{\left|\frac{dV}{dx}\right|}$ $\quad$ **Short wavelength limit.** Breaks down at classic/qntm transition. Linearize!

$\int_{x_1}^{x_2} dx\sqrt{2m(E - V(x))} = \pi\hbar\left(n - \frac{1}{2}\right)$ | $n \in 1, 2, \ldots$ no inf walls

$\int_{0}^{x_2} dx\sqrt{2m(E - V(x))} = \pi\hbar\left(n - \frac{1}{4}\right)$ | $n \in 1, 2, \ldots$ one inf wall - odd solns

$\int_{0}^{a} dx\sqrt{2m(E - V(x))} = \pi\hbar n$ | $n \in 1, 2, \ldots$ two inf walls

T.I.-N.Degen Perturbations (eg. quad Stark)

$E_n = E_n^{(0)} + \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \ldots$

$|n\rangle = \left|n^{(0)}\right\rangle + \lambda\sum_{k\neq n}|k(0)\rangle\frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}}$

| $V_{nk} \equiv \left\langle n^{(0)}\right| V \left|k^{(0)}\right\rangle \neq \langle n| V |k\rangle$, Non-std nrmlztn! $\quad$ Plrzblty: $\Delta = \frac{-1}{2}\alpha|\mathcal{E}|^2$

T.I.-Degen Perturbations (eg. lin Stark)

$E_l = E_l^{(0)} + \lambda\left\langle l^{(0)}\right| V \left|l^{(0)}\right\rangle + \lambda^2\sum_{k\notin D}\frac{\left|\left\langle l^{(0)}\right| V \left|k^{(0)}\right\rangle\right|^2}{E_D^{(0)} - E_k^{(0)}} + \ldots$ | D is degen subspace, $\left|l^{(0)}\right\rangle$ are e-states of $P_D V P_D$ | P_D is projector.

1. Sketch out matrix $P_D V P_D$. Dimen.s are degeneracy number. Square. Clmns/rows are distinct qntm nmbr states with degen.
2. Compute all matrix elements.
3. Find eigenvalues/vectors of matrix. Eigenvectors are the $\left|l^{(0)}\right\rangle$ "preferred states."
4. Write out matrix elements $\left\langle l^{(0)}\right| V \left|l^{(0)}\right\rangle$ etc.. for each $\left|l^{(0)}\right\rangle$.

Dyson Expansion

$$U_I(t) = \mathbb{I} + \left(\frac{-i}{\hbar}\right)\int_0^t dt_1 V_I(t_1) + \ldots + \left(\frac{-i}{\hbar}\right)^n\int_0^t dt_1\int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n V_I(t_1)V_I(t_2)\cdots V_I(t_n) + \ldots$$

T.D. Perturbations

$|\alpha; t\rangle_I = \sum_n C_n(t)|n\rangle_S$

$C_n^{(N)}(t) =_S \langle n| U_I^{(N)}(t)|q\rangle_S$

$C_n^{(0)}(t) = \delta_{n,q}$

$C_n^{(1)}(t) = \langle n|\left(\frac{-i}{\hbar}\right)\int_0^t dt' V_I(t')|q\rangle$

$C_n^{(2)}(t) = \langle n|\left(\frac{-i}{\hbar}\right)^2\int_0^t dt'\int_0^{t'} dt'' V_I(t') V_I(t'')|q\rangle.$

$V_I(t) = e^{\frac{iH_0 t}{\hbar}}V_S(t)e^{\frac{-iH_0 t}{\hbar}}$

$\langle n| V_I(t)|q\rangle = \langle n| V_S(t)|q\rangle e^{\frac{-i(E_q - E_n)t}{\hbar}}$

$|q\rangle$ is initial state. $|n\rangle$ is final state.

Fermi's Golden Rule (T.I. perturbation switched on at $t = 0$.)

Trnstn: $|q\rangle \to |n\rangle$

$C_n^{(0)} = \delta_{n,q}$

$C_n^{(1)}(t) = \frac{\langle n|V_S|q\rangle}{E_n - E_q}\left(1 - e^{i\omega_{n,q}t}\right)$

$prob(|q\rangle \to |n\rangle) = \left|C_n^{(1)}(t)\right|^2 = \frac{4|\langle n|V_S|q\rangle|^2}{(E_n - E_q)^2}\sin^2\left(\frac{E_n - E_q}{2\hbar}t\right)$

$prob(|q\rangle \to ?) = \int dE_n\rho(E_n)E_n\left|C_n^{(1)}(t)\right|^2 \propto t$ | ρ is D.O.S.

Symmetrization

para $\to$ spin singlet $\quad$ *ortho $\to$ spin triplet* $\quad$ Prob. Dens. $= |\psi_1(x_1)|^2|\psi_2(x_2)|^2 + |\psi_1(x_2)|^2|\psi_2(x_1)|^2 \pm$ Exchng. Dens.

Scattering

$\langle x| \psi\rangle = \frac{1}{L^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r}f\left(\vec{k}'_{out}, \vec{k}_{in}\right)\right]$ $\quad$ Scttrng Amp: $= f\left(\vec{k}'_{out}, \vec{k}_{in}\right)$ $\quad \left(\frac{d\sigma}{d\Omega}\right)d\Omega = \frac{\text{\# of particles scattered into } d\Omega \text{ per unit } t}{\text{\# of incident particles per unit } t}$

Dffrntl crs-sec: $\frac{d\sigma}{d\Omega} = \left|f\left(\vec{k}'_{out}, \vec{k}_{in}\right)\right|^2 = \frac{1}{\mathcal{L}}\frac{dN}{d\Omega}$ $\quad \mathcal{L}$ is Luminosity = # incident particles per unit A, per unit t.

Born Approximation

Incoming wave is **not** substantially altered by the potential. $\quad f\left(\vec{k}'_{out}, \vec{k}_{in}\right) = \begin{cases} \text{zero: } 0 \\ \text{first: } \frac{-m}{2\pi\hbar^2}\int d\vec{r}\, e^{i\left(\vec{k}_{in} - \vec{k}'_{out}\right)\cdot\vec{r}}V(\vec{r}) \end{cases}$ | $\vec{r}$ Centre of scatter.

Spherical symm: $f\left(\vec{k}'_{out}, \vec{k}_{in}\right) = \frac{-2m}{\hbar^2\kappa}\int_0^\infty r V(r)\sin(\kappa r)\, dr$ | $\vec{\kappa} \equiv \vec{k}'_{out} - \vec{k}_{in}, \quad |\vec{\kappa}| = 2k\sin\left(\frac{\theta}{2}\right)$

Basics

Free: $E_k = \frac{\hbar^2\vec{k}^2}{2m}$ $\quad$ Well: $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ | $n \in \{1, 2, \ldots\}$ $\quad$ Hdrgn: $E_n = \frac{\hbar^2}{2ma_0^2}\frac{1}{n^2}$ | $n \in \{1, 2, \ldots\}$